LECTURES ON THE LOGIC OF ARITHMETIC

By M. E. BOOLE

OXFORD: AT THE CLARENDON PRESS

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'Heaven lies about us in our infancy.'
'That true Heaven, the recovered Past.'

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TO LADY LOW

My dear Friend,

But for your resolute energy I think I should never have communicated to the educational world the method which I have long used for reviving the faculties of children suffering from mathematical rickets and logical paralysis. These diseases are common; they are induced by the practice of teaching mathematical processes on a hypothesis about the nature of Mathematics directly opposed to that which underlies the original invention and formulation of these processes.

I have, as you know, taken no part in recent attempts to improve methods of teaching, because, as James Hinton said, I can see no use in trying to invent good methods for teaching truth on a basis of falsehood; the only remedy in which I have any faith is to tell the pupil one or two simple laws of the relation of the human mind to Scientific Truth, and then to see that he forms the habit of working in accordance with those laws.

For nearly forty years, authorities of various kinds have been assuring me that it would be impossible to do this except in individual cases;

that the public will not tolerate being shown the glowing heart of the mathematical discoverer, will not dare to let itself know to what virgin inspiration it is paying homage when it confers medals and honours on an original mathematician.

Your Ladyship has intervened in the situation, refusing to see objections and difficulties, or to take 'no' for an answer. You insist that the public shall at least have the option of deciding for itself whether its children shall know what lies at the heart of genuine Mathematical Science. I must therefore ask you to accept your share of the responsibility of offering to parents and teachers what I fear is after all only a very feeble presentation of Arithmetic treated as a branch of the Art of Thinking, founded on the general Science of the Laws of Thought.

Yours affectionately,

M. E. Boole.

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PREFACE

The motto chosen for the title-page of this little work may seem unsuitable as an Introduction to a course of lessons on Arithmetic; a subject which to many persons seems so eminently un-heavenly and dull. But then, the main reason why it seems so is that their teachers failed to put them in possession of that 'Recovered Past,' the bearing of which on the Present forms the great clue to that knowledge of subtle forces which gives its possessor the key of the Future.

The present little volume is not intended to interfere with ordinary methods of teaching Arithmetic, or to supersede the books already in use. It can be used under any School system and in conjunction with any text-book. Not more than one chapter is intended for use in any one term; the earlier chapters are suited to little children, the later ones for children of fourteen or fifteen. The lesson for the term may be repeated each week with varying illustrations.

Teachers of such subjects as Electricity complain of the difficulty of getting pupils to apply what they know of Mathematics (at whatever

level) to the analysis and manipulation of real forces. It is not that the pupil does not know enough (of Arithmetic, or Algebra, or the Calculus, as the case may be), but he too often does not see, and cannot be got to see, how to apply what he knows. Some faculty has been paralysed during his school-life; he lacks something of what should constitute a living mathematical intelligence. In truth he usually lacks several things.

In the first place, though he knows a good deal about antithesis of operations (e.g. he knows that subtraction is the opposite of addition, division of multiplication, movement in the direction minus $-x$ of movement in the direction x, and so on), he has not the habit of observing in what respects antithetic operations neutralize each other, and in what respects they are cumulative; and surely no habit is more needed than this as preparation for making calculations in electricity or mechanics.

In the next place, he too often knows, about the idea of relevance, only enough to be foggy about it. The reason for this is that his study of the idea of relevance itself began where it ought to have ended; his attention was never called to it till the stage was reached when it would have been right that he should direct his action in regard to it sub-consciously from long

habit, leaving conscious attention free for dealing with the actual elements of some question which is difficult enough to need thinking about. He was not made to grasp the fundamental idea that *a statement may be relevant to one question and irrelevant to another*, till some knotty problem occurred involving consideration of which statements are relevant to the special question in hand. So he had to try to grasp at once the *idea* of relevance and the question, *what* is relevant to what, in a special problem. Such thrusting on the young brain of two difficulties of different kinds at once, is contrary to all accepted canons of Psychology. Examples are here suggested (Lesson XIII) in which there can be no doubt as to what is relevant to the question at issue; the child's attention is therefore free to focus itself on the idea that there can be facts concerning a thing which in no way concern the particular question which is just now being asked about the thing.

Then again, whatever skill he may have acquired in the manipulation of those notations and formulae which he has been taught to use, he knows hardly anything about the manner in which such things come into being. Now an applier of Mathematics to real forces should be able, when occasion requires, to modify his notation, or invent a new formula, for himself.

He cannot begin to learn how to do this, straight away, while his mind is struggling with problems of electricity or mechanics; he should have had, from the first, the habit of seeing through formulae and notations; of watching them coming into being, of helping to construct them.

The special difficulties found in interesting pupils in Arithmetic are in no way inherent in the nature of the subject; they belong rather to the domain of medical psychology; their cause can be fitly described only in the language of that science. However logical may be the course of a teacher's demonstration, *given his premises,* those premises are usually introduced to the pupil under conditions extraordinarily unfavourable to concentration of the attention-power. One can only describe them by saying that if analogous conditions were set up in the stomach-nerves at the beginning of dinner, the result would be lessened appetite and digestion-power; if in the muscles at the outset of a cricket-match, the running would in consequence be slower, the batting, bowling, and fielding less accurate, and the heart-action more irregular; if in the eye- and finger-nerves before a drawing-lesson, the result would be dazzled sight and trembling fingers.

It may be objected that if this temporary partial paralysis is set up, in the Arithmetic

class, only on the occasion of the first presentation of some new idea or principle, i.e. perhaps once in a term or two, it is not frequent enough to induce disease or permanent weakness of the attention-power. But if any condition of nerve disturbance or lowered vitality accompanies the *first* presentation of an idea or object, there is a strong probability that each subsequent contact with that object or idea will produce, by association, a recurrence of similar disturbed action. Thus is set up a habit of scattered attention which retards the development of the faculty of concentration, and makes any real grasp of principles difficult, especially in regard to Mathematics. One main object of the present volume is to present the premises involved in various arithmetical operations in a manner which concentrates instead of scattering the power of attention, and thus leaves the pupils fit to attend to the teacher's reasoning.

My own experience, however, is that a child who has been helped to grasp the premises of a mathematical argument, in any hygienic manner, seldom needs much assistance in making out for himself the logical consequences.

The Preface to a recently issued text-book of Arithmetic[1] asserts that 'the young are always eager for a rule or formula which will save

[1] *Arithmetic*, Kirkman and Field.

them the trouble of thinking for themselves.' The young who are reeling under a blow on the head from a bludgeon may not unnaturally be eager to be taken home to bed in the easiest vehicle at hand; and there is no telling what habits of lethargy might not be generated by a systematic course of such treatment; but the young who have been left in possession of their normal faculties prefer using those faculties in active personal exercise.

Another teacher says[1]:—

'In stocks and shares the pupil is usually dealing with things the nature of which he does not understand. A little political economy must be drilled into him. Bring the meaning of the abstractions of the subject home to him. . . . The great difficulty is in problems. Nothing but stern practice can avail here. Discourage all formulae. Anybody who knows anything about boys knows that if, by any means under heaven, they can avoid using their brains and attain the same result by a mechanical application of a formula they will do so. They cling to a formula as to a straw, in the ocean of new ideas in which they are drowning.' Exactly so; children's minds are too often left to clutch at straws 'in an ocean of new ideas.'

[1] Workman, 'Oxford Junior Local,' in *Educational Review*.

If, two or three terms before a pupil has to do with fluctuating values, he had any clear idea why a Bank of England note has a fixed value, it would be easier than it usually is found to make him grasp the meaning of stocks and shares.

The mechanism by means of which man learns is intended to play among the conceptions *Unity, Negation, and Fraction; Number is less cognate to it.* The kind of fractions dealt with in Arithmetic books is, it is true, too recondite to be thrust on a brain at an early stage. But if mathematical faculty is to develop normally in the youth, the child should have early practice in dealing with the idea of *a unit broken into bits, which fit together to make up a whole.*

Nature provides savages with opportunities for dealing with this idea, to an extent of which it is difficult for us to form any adequate conception. Something *has* or has *not* been caught for dinner; and when it has been caught it has to be shared and is entirely eaten. Next day the child again sees an animal of the same kind whole. The child's mind plays constantly between the idea *of rabbit* (or whatever animal may be used for food) *whole, rabbit shared, no rabbit; and again, whole rabbit.* This has a very different effect from what is

induced by the sight of a loaf or joint appearing as a matter of course; a loaf from which every one is helped, the rest being taken away when no longer interesting. This is one of the reasons why an existence uniformly, uninterruptedly prosperous is not always the most favourable to intellectual development.

In all ranks of life, it is true, infants provide themselves with opportunities for exercising the faculty of reconstructing a unit from its fractions, by tearing or breaking things and fitting the bits together again. Our Kindergarten systems to some extent provide for systematic exercise of the same kind by means of various toys. Even there, however, though much practice is given in constructing, too little is given in re-constructing a unit out of its fractions. But in schools the whole provision for this practice, which is the essential fibre of all mental growth, seems left to chance, and too much crowded out.

In Arithmetic, where it is specially important, hardly any opportunity is afforded for practice in swinging the mind between the conceptions Unity, Negation, and Fraction. The mass material of Arithmetic itself (i. e. the art of dealing with numbers) is packed into the mind artificially; little or no possibility is provided for it to build itself up by natural accretion

round its organic supporting fibre. This tends to induce a condition of what I have called mental rickets.

It is desirable to fix children's minds on the act of Negation as a positive act of mind, to a far larger extent than is usually done. Opportunities for this might be taken in connexion with sums in multiplication and long division; the zeros should be written in, as statements of fact, for some time after the pupil has grasped the idea that their omission does not actually affect the ultimate answer.

Use might also be made of such exercises as those of Chapter XIII.

Again, the teaching of Arithmetic is much facilitated, if, besides actual exercises on 0 and 1 and very simple fractions, logical exercises parallel to the operations of Arithmetic and constructed on the same models, are carried on in that borderland region where mental operations parallel to those involved in Arithmetic deal with questions of Art, or such simple portions of History, Ethics, or social relations, as come fairly within the scope of a child's intelligence. Each operation of Arithmetic may find its analogue in one of these borderland exercises in Logic.

It is of primary importance that children should clearly distinguish between what they

do or recite because they are told to do so, and what they themselves grasp or see. They must do certain things because they are told, and commit to memory facts which they will need to know; but they should always be taught to distinguish between :—

'I repeat this because I was told,' and 'I see that this is so.'

Some of the worst mental habits are induced by the practice of teachers making a statement as if *ex cathedra*, and then proceeding to bring forward proofs of its truth; this is Euclid's method, but Euclid apparently wrote for grown-up men, perfectly well able to take care of their own minds; he wrote probably for the best intellects of his time. His method is unsuited to children. If a teacher has anything to say to children as a statement, he should say it, not exactly as dogma which they are bound to believe, but as working hypothesis which they are to assume as a basis for the present. Anything which he intends to prove should never be stated; children should be led up to find it out for themselves by successive questions. No pains should be spared to keep the two sets of statements apart.

To return to Mr. Workman's complaint of the abuse of formulae, surely the remedy lies not in discouraging the use of them, but in

encouraging children to use them intelligently and to make them for themselves. They should begin early the practice of entering certain kinds of results in a book. This book may be divided into two parts: anything specially needful to remember, if found out by the children, should be entered in one part; if told them by the teacher, in the other.

The sentimental people who assert that everything in Arithmetic can be 'proved' to children have, usually, no idea of what rigid proof means; it is not necessary that the child should see the evidence for every hypothesis on which he works; what is necessary to mental health is a clear understanding of what constitutes evidence, and the power to distinguish between what is, and what is not, proved.

The children should be encouraged to use freely, for reference, any tables or formulae constructed by themselves. This should be permitted as an indulgence, a labour-saving luxury, the reward for accuracy in recording the results of previous investigation. In private teaching and in very small classes, no child should be allowed to enter any formula into the form-book till he is individually able to work it out unaided. In ordinary class-teaching this may be impossible. In that case the process should be gone over at intervals, till

the class as a whole can, amongst them, evolve every step. The formula should then be entered by each child in his own form-book; and the impression should be conveyed that it is the communal work of the class, the expression of its conjoint logical investigation.

The formula-book should begin with a blank form of multiplication-table, which is to be filled in by the child as practice in addition. The operation of multiplying should be introduced to the child's mind in some such way as this:—'Add together four and four and four and four and four.' When the addition sum has been done correctly, say:—'You will have to add five fours together very often. You may enter the result in your table-book; and next time I give you five fours to add together, I will allow you to look for the answer in your book instead of adding.'

All the modern higher Mathematics is based on a Calculus of Operations, on Laws of Thought. All Mathematics, from the first, was so in reality; but the evolvers of the modern higher Calculus have known that it is so. Therefore elementary teachers who, at the present day, persist in thinking about Algebra and Arithmetic as dealing with Laws of Number, and about Geometry as dealing with Laws of Surface and Solid Con-

tent, are doing the best that in them lies to put their pupils on the wrong track for reaching in future any true understanding of the higher Algebras. Algebra deals not with Laws of Number, but with such laws of the human thinking machinery as have been discovered in the course of investigations on number. Plane Geometry deals with such Laws of Thought as have been discovered by men intent on finding out how to measure surface; and solid Geometry with such additional Laws of Thought as were discovered when men began to extend Geometry into three dimensions. The branch of Mathematics called Quaternions deals with such Laws of Thought as reveal themselves during the process of investigating the relations between n and $n + 1$ dimensions. The sooner pupils are made to see all Laws of Arithmetic as Laws of Thought, not of things, the simpler and more satisfactory will their future course be.

This is especially the case with regard to the sign $-$; it should never be interpreted in any such way as to convey the impression that it indicates negative quantity, subtraction, or diminution of numbers or of things; but always as indicating something about the point of view from which the things or numbers are considered. Neglect of this caution lands one

in endless metaphysical contradictions and absurdities. Such for instance as that a rectangular surface becomes = o if we happen to place the origin of co-ordinates in the middle of it; for it then divides naturally into the four rectangles $X \times Y$, $X \times (-Y)$, $(-X) \times (-Y)$, $(-X) \times Y$. The amount of confusion and hindrance to progress thus induced is sickening to think of.

Teachers seldom observe how early many children are affected by such anomalies as these, or how profoundly they are affected by them. If the same boys who are thus painfully impressed became teachers in their turn, they would no doubt remember their own past experience, and a better mode of presenting the primary conceptions of Arithmetic and Geometry might soon be evolved. But, unfortunately, these logical thinkers are usually repelled by the impossibility of extricating themselves from the network of insincerity, or what seems to them such; give up the study of Mathematics, and betake themselves to some other pursuit. Very much of the waste and confusion which gather round the sign $-$ might be obviated by making children go through such exercises as those given in Chapters VIII and IX.

Much needless difficulty is caused by the attitude of Mathematical teachers towards the

symbol ∞, which they often interpret as somehow connected with endless length or exhaustless quantity. The effort to conceive of either endless length or exhaustless quantity is a metaphysical gymnastic very unhealthy for a young mind. It sets up also a set of ideas and connotations quite irrelevant to any use of the symbol ∞ in the higher Mathematics. ∞ has reference not to length or quantity, whether great or small; but to release from certain restrictions to which the values specified as finite have been subject. (See Lesson XV.)

The whole terminology connected with the operations called in Arithmetic books 'Multiplication' and 'Division' is excessively misleading. The words 'multiply' and 'divide' might have been invented on purpose to create confusion in passing from integral to fractional Arithmetic.

The continued use of the word 'dividend' especially, in Arithmetic, is a symptom of the carelessness of teachers in the matter of avoiding causes of confusion. In old days, when most people who had money hoarded it in the house or deposited it in a bank, till they could use it in some business of their own, the word 'dividend' had no connotation for a child's ear, till it was explained to him as 'an amount *to be* divided.' But in these days of limited liability companies, the word 'dividend' is incorrectly but very

commonly used, at home in the hearing of children, to indicate a sum of money which is the result, or quotient, of the division and distribution of the year's profits among the shareholders: 'I will do so-and-so when my dividends come in,' &c. Even if the teacher is careful to explain that the word in the text-book does not mean what it means in common parlance, which not all teachers remember to do, that explanation in itself causes distraction; some word should be chosen which has no misleading associations. Of course children 'outgrow' any misapprehension which may be thus caused, and learn that 'dividend' has two meanings. But one wonders what teachers suppose is the effect on the mental habits, on the brain tissue, of struggling, for even a few days, to understand division before this discovery has been made! The present writer has not authority to devise a nomenclature; and therefore in this introduction employs that in common use, though under protest. It should be avoided in speaking to the class.

The teacher should, in dealing with any branch of the operation called 'multiplication,' have in his mind this idea:—'Multiplication' means doing to the operand (multiplicand) what, if done to unity (1), would produce the operator (multiplier).

In dealing with 'division' he should keep in his mind the idea :—'division' means doing to the operand (dividend) what, if done to the operator (divisor), would produce unity (1).

The operation of division should be introduced at first, not as the inverse of multiplication but as a succession of subtractions, thus :—

If five boys are to share twenty apples among them, how many can each boy have? Let us give each boy one apple first; that takes off? Five apples. And leaves? Fifteen. Now we give each boy another apple: that takes off? Five more. And leaves? Ten. Another to each boy leaves? Five. Another to each leaves? None. So now we have finished. How many times did we go the round and give each boy an apple? Then each boy has? Four apples.

A little practice in such successive sub tractions should be gained, before the idea is introduced of doing long division straight off. Division should not be treated as the mere inverse of multiplication, but as a way of cutting short a laborious series of subtractions. The child should be led to see that the multiplication table can be used to cut short not only series of similar additions, but also series of similar subtractions. A great deal of the difficulty afterwards found in dealing with such

operations as G.C.M. is traceable to neglect of this simple precaution.

The manner in which that most logical of all text-books, Euclid, has been misused to induce illogical habits of mind, repeats itself in most departments of elementary Mathematics. A good instance of such misuse can be shown in connexion with the subject of equivalent fractions (Lesson XVI). Let us suppose that the teacher wishes to prove to the class that three-quarters of one is equal to one-quarter of three. He takes as his unit of thought the concept apple. He too often begins by stating a thing which is not true in itself: viz. that three-quarters of an apple is the same thing as a quarter of three apples. For the purpose with which the teacher's mind is occupied at the moment (the equivalence of fractions), the statement is true; it is as true as that, from the shop-keeper's point of view, twelve pence are a shilling. But the child, who has not yet been introduced to the conception of equivalence of fractions, is at a point of view quite different. To the child's imagination 'great lots of grub' have a fascination quite independent of the size of his own share. To suggest (for instance) that a bun for one child is the same as a hundred buns for a hundred children would be to insult the pupil's understanding and his feelings—

feelings which affect him all the more keenly that as yet he does not know how to express them. If he *could* believe you, what would be the meaning of school treats and social picnics? In proportion as you ultimately induce him to feel that one bun for a child is the same as a hundred buns for a hundred children, he will be, in future, the worse man, citizen, political economist; and the less fit to apply mathematics to problems in real forces. But we are speaking now of the effect of such statements on his intellectual processes, at the age when his sensations are as yet unwarped. The statement that a quarter of three apples is 'the same as' or 'equal to' three-quarters of an apple has brought prominently into the field of his imagination two pictures:—an apple, and a group of three apples. The natural way of making an apple equal to three apples is to bring forward two more apples. It is the way things used to be made equal when he was learning addition; why not now? The teacher has therefore started him looking out for the two more apples; and as what is said next does not seem to be tending in that direction, his attention is distracted; he gives only half his mind to what is said. Thus he fails to get hold of one or more links in the chain of argument. Some children soon recover themselves, and

attend to the rest of what is said. And, as the teacher seems to be satisfied with the reasoning and to expect them to be so, they imagine they are so. This helps to form a habit of intellectual dishonesty, of confusing sham proof with real proof. The children who are either less sharp or more logical and thorough, look longer for the non-appearing two apples, and miss more links of the chain of proof. These children feel that the whole demonstration has passed in that region where grown-ups conduct a self-satisfied mental life in which children cannot share. Thus are formed habits of 'hopeless non-comprehension,' perhaps even of 'self-protecting and contemptuous non-attention.' The chapter on equivalent fractions (Lesson XVI) suggests ways of attacking the problem not open to this objection.

It may occur to some that too much use is made of examples relating to food. But we cannot make mathematicians by insisting upon a non-existing superiority to physical facts. Apple or bun forms the natural unit for a child; the *sharing* of a cake or fruit is the natural fraction as well as the true introduction to the higher ethical life. As a matter of fact the Arithmetic of grown people is largely occupied over questions of food supply and of personal or family interest. To place children's ideas about

such subjects on an honest basis and give them a social direction would surely tend more towards ethicalizing the community than the setting up of topics of fictitious interest, and then teaching children to be satisfied with reasoning which is either imperfect in itself or but partially understood by them. To let the children be thoroughly logical about their own sphere of action among things which they do care about, and deal practically with duties which they must perform, is far better training than to encourage them to express opinions about subjects the data of which lie beyond their personal ken.

The pauses of sitting with slack muscles and taking slow quiet breaths, to make mind-pictures, is not intended to supersede the usual permitted 'intervals' between lessons; they are an integral part of the Logic lesson itself. They, as well as the rest of the scheme here suggested, form, it is believed, the first attempt to adapt to elementary education here the magnificent method of study described in Gratry's *Logique.* A certain element of normal mental life, essential to sound intellectual development, and which can only be received during suspension of the active perceptions of intellect, is being driven out by modern educational grind. This tends to produce a sort of mental and moral rickets, analogous to the physical disease induced by

a diet deficient in the bone-forming element. Teachers who have studied any sort of medical psychology bitterly lament that the requirements of existing systems force them to inflict this grievous wrong upon the rising generation. In the present volume, an attempt is made to re-introduce at least a small share of the skeleton-forming elements of mental life, in a manner which does not necessitate disturbing existing arrangements. The author has been kindly assisted in this matter by Mrs. Archer and her pupil Miss Cross, of Coombe Hill School.

English children have now to be taught to relax, to breathe, to see. It is probable that the cultivated art of relaxation will ultimately prove more satisfactory than the mere following of natural impulse to relax.

But it is certain that the teachers who have weeded out the wild growths must see about planting the cultivated ones. Such exercises as those here recommended tend to promote sane and healthy mental action, and to make the discipline of school more organic and harmonious. They have a great fascination for children, as my young friends at Coombe Hill School know. I am sure they will be glad that I am attempting to give to other boys and girls a share in the fun that we had over our little sums.

Mary Everest Boole.

INTRODUCTORY ADDRESS TO YOUNG TEACHERS

Some of you may perhaps have seen what are called Japanese Flowers. You buy a penny packet of things which look like rather clumsy wafers, each of which has a fine line round its outer edge. You put one of the wafers into warm water, and (as the printing on the envelope says) 'Watch the result.' The line round the edge melts and slips off. Then perhaps, from the wafer come out first one little coloured flower, then another, and another. Or there may come out leaves, or sea-weed. The change seems wonderful, but the explanation of it is simple enough: some one in Japan, by great care and skill, made each leaf and flower; then they were linked together by stalks made of thread, and then the whole set was rolled up into a tight form convenient for carrying about, and fastened round with a little strip of paper. And we have to put them into warm water which acts as a *solvent* (that is to say, a melter), before we can see how pretty they are.

Arithmetic seems to some people dry and un-beautiful; but that is because they have not soaked it in that solvent which is called sympathy. If we had sympathy with the struggles and labours of others, Arithmetic would be easier to understand and pleasanter to learn than many children find it.

It must, however, be distinctly understood that any of the following lessons are to be omitted or modified, if, in the judgment of the teacher, it is unsuited to the condition of the class.

This remark applies especially to Lesson I. If the discipline of the class is in such condition that the

teacher would not find it easy to restore order after a burst of laughter, he may suggest to the children, in some quieter way than that here given, that men held up all their fingers, as a way of saying 'Many,' before they had learned to count how many. But my own experience and that of several of my friends is that the dramatic introduction of the savage element, where this can be safely permitted, has a wonderful effect in conquering the apathy of which so many teachers complain, and making Arithmetic real and living. This remark applies to children of all ages, and at all social levels.

In this book you will find no long sums. The following caution may be found of use in dealing with those which you set from the ordinary text-books. Children should be carefully taught to suspend attention at proper pausing-places; as, for instance, at the ends of lines, in multiplication, long division, practice, or equations. Just as, in reading aloud, one must take breath at suitable stopping-places, or one will be driven to stop for breath involuntarily at the wrong place, so it is with the attention-power in Mathematical work: if it is not trained to take a moment's rest at the right place, it rests itself (or, as we say, 'the mind wanders') in the middle of a line of figures; a wrong figure is written down during the lapse, and sets the whole sum wrong. This acts very unfavourably on those children who have specially short-breathed attention-power; who are often, otherwise, the cleverest, and might become, if properly trained, the most intelligent mathematicians. On the other hand, a training in the proper care of the attention-power is good for all children alike.

I

HOW MEN LEARNED IN OLD TIMES

You have heard, my dears, that, a long time ago, there were no people in this country except savages. Perhaps some of you have read that delightful book, *The Story of Ab*; or you may have seen the picture of a savage with his wife and baby, at the beginning of Clodd's book, *Primitive Man.*

How do you think children learned lessons in those days? Do you think Ab's wife ever told little Mok to wash his face and hands[1], and come to lessons? Do you think that the hairy woman in Clodd's book ever called out to her boy: 'The school bell is ringing; make haste, or you will be late'? Oh! no; nothing of that sort

[1] [In old Miracle Plays the first woman is represented as calling her boys in to wash their hands and brush their hair, because to-morrow will be Sunday and they will be examined on the Catechism. The future villain of the piece initiates his evil career by some such breach of good manners as taking off his cap with his left hand! These and countless other literary phenomena of the same kind go to show how far astray the uninstructed imagination goes in its presentation of historic data; and how necessary it is to take frequent opportunities of pulling it into line with the reality of things.]

ever happened at that time. Yet the children did not grow up quite ignorant. We are going to have a little talk to-day about how people learned in those days, and what sort of teachers they had to make them learn.

Why do the monkeys in the Zoological Gardens like climbing on their ropes? Why is it found good for the health of children to climb the gymnasium ladder and cling to the bar by their hands? Why do most of you enjoy doing so as soon as you have got a little practice at it? Why do dogs not climb ladders or ropes, and why is it not found good for their health to teach them to do so? You and the monkeys had ancestors who climbed trees in order to be out of the way of wild beasts of the dog kind. These ancestors of yours forced themselves to learn to climb, by great labour and effort, driven by great fear and desperate need. But they went on, generation after generation, trying and trying, till it came quite easy and pleasant, till they could climb for pleasure and fun. And that has worked into the very muscles of your flesh and the blood of your veins, the power and the wish to climb. But the dogs seem to have had no ancestors who learned to climb; when their ancestors were afraid of enemies, they protected themselves in some different way. Primitive Man climbed trees quite easily, and

so did Ab, and Ab's wife, and Mrs. Primitive Man; they carried their babies up trees till the babies were old enough to learn how to go up; and then the mothers made the children learn. And all that has helped to make climbing natural and good for children now.

But another thing was happening at the same time. Have you noticed what sort of words a child learns first? Names of *things*, or names of *numbers*? For instance, if a child of about a year and a half old saw six birds on a wall, would he say 'Six,' or would he say 'Dickie-birds'? You know that he would say 'Dickie-birds' much earlier in his life than he could count 'six.' Well, savages of very long ago, such as Primitive Man, could not talk as we do; indeed, one name for them is 'Speechless Man.' But they must have had words or grunts or growls or signs of some sort for some few things; even a hen can tell her chickens that she has found food, or that she sees a hawk. We may be sure that Primitive Man could tell his wife that there was a wolf near when he could not say 'Six wolves.' Perhaps he said 'Woo-oo-oo-oo' by way of a sort of imitation of a wolf's howl.

And presently he began to find out that it would be convenient if he could somehow tell his wife whether there was one wolf or

many. He had no words, as yet, to tell her with.

Now what does an omnibus conductor do, when you have heard, or guessed, that he said 'Fares, please!'; but, because of the rattling noise, he cannot make you understand whether your particular fare is two pence or three? He holds up fingers. Can we not fancy Primitive Man doing that? If he said his word or grunt for wolf, and held up one finger, that might mean: 'Just stand behind me with baby, my dear, while I kill the wolf with a stone.' But if he spread out all his fingers, that might mean: 'Let us get up the tree quick; for there are more wolves coming along than you and I can manage.'

We see now that the savages of long ago had teachers; though they had neither desks nor books nor school bells. Outside of themselves there were wild beasts which taught them to be sharp and try to escape danger. Inside of them they had feelings which taught them that it is not always enough to climb up a tree by oneself, that they must help each other and take care of the baby. And, between the two sets of teachers, they somehow learned to say 'One' and 'Many' on the fingers of their hands. Neither set of teachers could have taught them alone; neither the wolves outside them nor the

feelings inside. But between the two sets, the savages learned their lesson so well that, to this very day, we have never forgotten it. Children count on fingers still, unless there is some reason for using counters instead.

Poor Mrs. Primitive Man, and all the rest of our wild ancestors! I often think of them when I am puzzled and worried myself. They little guessed what delightful gymnastics they were preparing for you children, or what a convenient way of doing sums they were helping to prepare for us all, by their care of their babies and by their brave struggles against laziness and cowardice. Perhaps, thousands of years hence, people will be enjoying some wonderful science or delightful art, of which we now have no notion at all; and will be looking back with thankfulness to the struggles and troubles of us now, which are preparing for their enjoyment.

Now before you go to the next lesson and forget this one, I want you to make a picture in your minds that you will remember in after life.

Shut your eyes; lean back comfortably in your seats. Let your hands lie quite slack on your laps. Take a few long easy breaths. Now then.

(The Teacher should read the following with a pause between each sentence.)

Think of the funny hairy man in Clodd's book. See him start suddenly. He calls out: 'Woo-oo-oo-oo.' The little ones may call out 'Woo-oo-oo-oo' if they like. Now be quiet again. He calls out 'Woo-oo-oo,' and puts up his hands with all his fingers spread out. His wife comes out of her dreamy mood. Jumps up and carries her baby to the foot of the tree. Clasps the baby between her long hind feet. Climbs the tree like a monkey by her hands only. The man climbs after her. When they have reached a high branch, she takes baby in her arms: she and the man sit side by side, and wait till the wolves have gone by.

(Two or three minutes should now be spent in silence, sitting at ease, before the class breaks up. Attention should be paid to relaxation of that muscular tension which accompanies all active attention to external facts.

Modern children are fast forgetting the secret of slow, deep natural breathing, as well as that relaxed attitude and meditative picture-soakage which is one of Nature's most powerful educators. The loss of these things is probably one great factor in the failure of modern educational schemes.

As the sudden rush from active attention to one topic to active attention to another is in itself bad for the young brain, the last few minutes of each class seems a good opportunity for practising concentration.

The above method for impressing mind-pictures on children is an adaptation, suited to ordinary class teaching, of a more elaborate method carried out at Coombe Hill School.)

II

ON COUNTING BY TENS

BLACK-BOARD.

WHAT is this, **1**? And this, **2**? And this **3**? (and so on), and this, **9**?

Now I want to write ten: how shall I do it?

Put **1** and **0**.

But what has ten done to be different from the rest?

Why should it have two signs instead of one like its neighbours? and why does it take signs belonging to its neighbours, instead of having one of its own?

Did ten ask to have two signs? Did it wish to have two? No; then why do we give it two?

I once asked a young friend of mine why he did something in his sum; and he answered: 'My reason is that I was told to do it at school; but I know I ought to have another reason, and I know I haven't.' I thought that was a sensible answer. It applies to most things in arithmetic: your reason for writing ten with two signs is that your teachers told you to do

so; their reason for telling you to do so is that it has been found a convenient and suitable way for people to write ten. But there is a reason why it is suitable and convenient, and you ought to know that reason. All your sums will seem to you more interesting, and, I think, easier, when you know it.

We have seen under what sort of pressure people must have found out how to tell each other that there were *one* or *many* wild beasts. But when once they had learned that much, they could apply it further. By-and-by they began to keep flocks of sheep or goats, and they made stone walls to fence in the sheep by night, so that wolves could not get at them. Perhaps they might wish to count their flocks, to see if all the animals were safe in for the night; or one man might wish to sell his sheep to another. Well, at first the owner might count by putting up a finger for each sheep, as it passed in through the gap in the fence. That did very well up to ten. But what next? How could he show anybody that there were sixteen sheep or twenty-nine sheep? He might put up all his fingers for ten, and then put them down; and then put up six fingers for the other six sheep, that would mean sixteen. He might put up all his fingers twice for two tens, and then nine fingers; that would mean twenty-nine.

But if he had many tens of sheep, he might lose count of the number of times he put his hands up and down. What was to be done about that?

The man who had very many sheep to count must contrive somehow to make a mark for each time he put his hands up.

There was no paper in those days, no pens or pencils, not even balls strung on wires such as we see in some infant schools. I believe people had not even names for more than a very few numbers. There are still people in the Malay Islands who have no names for any numbers except one, two, and three. People had to think about keeping flocks in order to be sure of having food and clothes for their families, before they began thinking about what was the best way of doing sums! What they wanted was to make sure somehow that all the sheep they sent out in the morning were safe inside the fence at night. There was a quicker way to manage that than inventing figures, and finding out how to make pencils. The man could call up his little daughter, who had no school to go to, and say to her: 'Would you like to help Daddy count the sheep? Just stand nice and quiet beside me, and I'll tell you presently what to do.' Then each time he finished putting up all his fingers, he would

make her put up one finger, till all the sheep had passed out. Then he would look well at his own fingers and the little girl's, and make a picture on his mind of how they looked. Then he would perhaps say: 'Now, dear, when the sheep come home we must get our hands looking just like this; and we must not be satisfied till they do.' In the evening, if his hands and hers looked as they did in the morning, he could take her in to supper, and have a game of play with her, and then go to bed quite happy, because all his sheep were safe in for the night. But if either he or she had too few fingers up, and no more sheep seemed coming in, he knew that he must send her in to her mother, and call his big son, and go out on the moor to look for the rest of the sheep.

Now I think we had better spell a number in the way that the shepherd and his child did; then you will know better what sort of way it was. Tom, come here please; stand at the right hand of the class; you shall play at being the shepherd. Jane, you stand at the left side of the class; you shall play at being the daughter. I cannot let the rest of you rush about pretending to be sheep, in school; you can do that in the play-ground if you like; but, in school, I will say 'sheep,' and you children must try to make a mind-picture of sheep going

out through a gateway, one by one, as fast as I count. Tom, every time I say 'sheep,' put up a thumb or finger. Jane, wait till I tell you what to do. Now, Tom :—

※ One sheep; two sheep; three sheep; four sheep; five sheep; six sheep; seven sheep; eight sheep; nine sheep; ten sheep.

Now you have no more fingers to put up. Hold your hands up high, and show the class that you have no more fingers left to count with. Put your hands down. Jane, put up one finger; that means that Tom has put up all his fingers once. Hold it high up for a minute, for the class to see that you have one finger up. Now, you need not keep your hand so high any longer, it would make your shoulder ache; but keep that finger out steadily all the time till I tell you to leave off. Now Tom :—One sheep; (repeat ※).

Hands down, Tom. Jane, put up one more finger to show that Tom has gone over all his fingers once more. Tom, one sheep; (repeat ※).

Hands down, Tom. Jane, one finger more. Tom, one sheep; two sheep; three sheep; four sheep; five sheep; six sheep; seven sheep.

Now, both of you, hold your hands high, so that the class can see them. How many fingers has Jane got up? Three. And each of

the fingers means? That Tom has counted ten on his fingers. So Tom has counted ten three times; how many does that make? Ten three times? Thirty. That is right. But Tom has done something else besides putting up ten fingers three times; he has also put up seven fingers besides. How many sheep has he counted then? Thirty-seven. We will pretend that this is early morning and the sheep have just gone out to feed. Look well at the hands. When the sheep come home at night, the hands must look just so, or else the shepherd cannot go happily to bed; he must go and see where the missing sheep are.'

If you play that game out of school, you can play at counting other numbers; forty, fifty, seventy, ninety sheep. But that would take too long now.

For many, many generations people had only fingers to count with; and, all that time, one finger of some other person may have stood for ten fingers of the person counting. So, at last, when people found out how to write, it *came natural* to them to make one in the left-hand place of the paper stand for ten in the right-hand place, just as it comes natural to monkeys and boys to climb things, because their forefathers had done it for so long that they did it without much thinking about it. It seemed easier to

do it that way than to think about any other way; the inclination to do it so had got worked into the very marrow of their brains. And so, when they came to want to reckon more than ten tens, they put the figure for ten tens in a fresh place; and that is what we call the hundreds' place of a row of figures. And so they went on and on, making fresh places for the different *powers* of ten (as some people call it) because their great-grandfathers had ten fingers to count with.

Now we come back to the questions we started with just now. Why does ten have two signs belonging to other numbers instead of one sign of its own? It does not; it has one sign. Its sign is a '**1**,' *written in the tens' place*, and that means: once the hands are put down, after counting ten, in order to begin counting over again. When we write twelve, which means ten-two, we write it so '**12**'; that is, one ten or one double-hand, and two in the place of units as we call them. But if there are no things over, no units, we say so; we put **0** in the units' place. That helps to remind the writer, and to show to other people, that the **1** stands for one ten, not one thing.

If ever you are teaching a baby brother or sister to count, there are three things I should like you to be careful about. Never teach

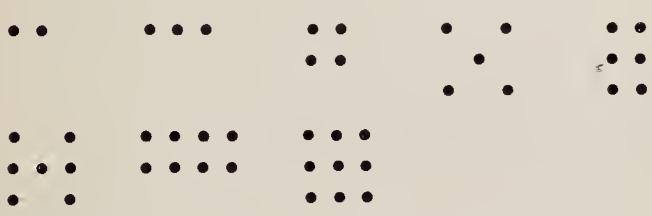
a baby to say 'one, two, three,' like a parrot; always teach him to count *things*. If you have no counters, use some sort of blocks, or bricks, or collect little pebbles, or buttons. (Only then you must be careful not to let him swallow them.) Or he might count your fingers and toes, or the railings. But it is best to have things he can shift about, such as buttons; and let him push each, as he counts it, from the uncounted heap to the counted heap. Never make a little child say 'eleven, twelve'; always tell him to say 'ten-one; ten-two; ten-three; ten-four'; grown-up people say 'eleven, twelve,' and now you are old enough to go to school, you must do the same; but a little child should always say 'ten-one; ten-two'; like 'twenty-one, twenty-two.' And as soon as he has counted ten things, make him put them in a little box or in a heap by themselves; that will make him understand what he is about much better than most little children do. And then you can tell him the stories about Mr. and Mrs. Primitive Man, and the baby, and the wolves; and about the shepherd and his little daughter.

III

WHY WE DO NOT ALWAYS DO SUMS THE WAY THAT COMES NATURAL

It is often best to do things the way that 'comes natural,' as long as there is no reason for altering it. Very many things have to be altered as we become civilized and have to live in towns. It comes natural to children to count on their fingers, but they have to be made to use counters or blocks instead, because counters can be arranged in convenient patterns:—

Black-board.



Whereas fingers are much less convenient.

It came natural to bees to make round cells, because their forefathers had made little holes by rubbing their bodies round; but when the bees took to living a great many of them in one

hole, they had to be stopped doing what was natural, and made to learn to fit their cells into each other, because it saved room and was more convenient. Poor little bees! They didn't at first like learning the new way of making cells, I dare say; but they had to. It comes natural to you to do some things that have to be stopped because they are inconvenient. It comes natural to a baby to grab with its hands at everything it wants; but it has to be taught to eat properly with a spoon. There are such quantities of things that we must alter, that it is generally best to go on doing things in the old way until a reason comes for altering them. It would take a good deal of trouble now to learn to count ordinary things otherwise than in tens. There is no particular reason why we should do so, so we all go on counting by tens; and you were made to learn to do the same. Half the use of school is that children should learn which things they had better do the way that their forefathers did them, and which things we must now learn to do in some other way.

Once upon a time a dear little girl about a year old was sitting on her mother's lap, and a bright teapot full of hot water was on the table close by. It comes *natural* to babies to try to get hold of anything bright that they

see. So this baby put out her hand towards the teapot. Mother said 'No,' and pushed the little hand away. But baby thought she *would* do what was natural; she screamed and put out her hand again. Mother began to think that if baby was so anxious to get at the bright things, she might make a grab at the candle or the lamp some evening, or crawl to the fire, and have a bad accident. So she thought she would let baby have her way at once, when it could not do any real harm; she pretended to give in, and let baby touch the teapot. It was very hot. It did not actually injure the child. But somehow, after that, baby understood why it isn't always nice and comfortable to do just what comes natural!

You know that we must do sums in certain ways because we are told to do them so; but also we ought, when we can, to have another reason. It is a good thing to play sometimes, when it does not matter, at doing a sum the 'wrong' way, the upside-down way to what we have been taught; then we find out why we were taught a way that does not seem natural. We are going to do that to-day.

Suppose I have in my purse a shilling and two pence; and some one pays me two shillings and three pence, how shall I write the account of what it ought to come to?

BLACK-BOARD.

s.	*d.*
1	**2**
2	**3**

'Three pence and two pence are . . . '

Stop a minute. Which does one care most about, the pence or the shillings? The shillings. Well, why don't we reckon the shillings first? 'It is the wrong way.' Why is it the wrong way? Would it make the sum come wrong?

(Usually some children in a class think it would make the sum come wrong.)

Well, we are only playing to-day, it does not matter if the sum does come wrong. Let us try.

One shilling and two shillings make three shillings. Two pence and three pence make five pence.

s.	*d.*
3	**5**

That is the answer we get by beginning at the shillings end. Now let us try beginning at the pence end, as we are told is the right way at lessons. Two pence and three pence make five pence; one shilling and two shillings make three shillings. So we have five pence and three shillings. Does that come to the same as

three shillings and five pence? Yes. And ‘three and five pence’ is more natural to say than five pence and three shillings. Then why are we taught to do the sums in the way that is not natural, since both ways come to the same answer?

We must find out if we can. We must know the real reasons for things when we can. Let us try another sum:—

BLACK-BOARD.

s.	*d.*
2	6
3	8

Three shillings and two shillings make five shillings. Eight pence and six pence make fourteen pence. How do we write fourteen pence? Two pence and one shilling. Where must we put the one shilling? Along with the five shillings. So now our sum looks like this:—

s.	*d.*	(*a*)
2	6	
3	8	
$\cancel{5}$	2	
6		

or else like this

s.	*d.*	(*b*)
2	6	
3	8	
5		
1	2	

Now let us do the sum the way we are shown at lessons. Eight and six are fourteen. Put down two pence and carry one shilling; one shilling and three make four; four and two make six. We have our sum looking so:—

s.	*d.*	(*c*)
2	6	
3	8	
6	2	

※ Now do you see why you were told to do addition sums in a way that is not the natural way? Fancy if a shopman gave us our bills all crowded up with scratchings out and extra figures at every turn, like (*a*) or (*b*)! It is necessary that you should learn, while you are young, to do some things in the way that you would not think natural. Your teachers pick out for you which way you must do them in class; they have experience and know; but in holidays you may always try doing easy sums the upside-down way to the way you are told in school,

so as to gain experience for yourselves, and find out the reasons of the school rules.

It is not safe to experiment, while you are young, with real things—with fire or food or your own bodies, because you might do harm that could not be undone. But sums can be wiped off or scratched out, when you get them in a muddle; so they are capital things to experiment with, and great fun too; and you will understand your real work at school much better if you experiment in play. Always do your school-work, in term-time, exactly as your teacher tells you; that is the way to grow smart and handy and useful. In holidays, do easy sums the upside-down way to the school way. That is the way to understand your work, and grow clever enough to find out things for yourself.

At ⁘ the teacher may interpolate other examples of beginning sums at the wrong end, taking an easy example of subtraction, multiplication, and short division.

Look well at the black-board. Now shut your eyes and settle at ease.

Can you see in your minds the sums (*a*) and (*b*)? (and any others that may have been done in roundabout ways). If not, open your eyes and look. Shut your eyes again. What do you see?

IV

ARITHMETICAL SHORTHAND

Black-board.

12 Buttons are a dozen.
12 Pence are a shilling.

Are those statements true?

Are they both true?

Are they always true?

Are they true in the same sense?

Let us see.

Suppose you go into a shop and say: 'I want a dozen of those buttons, please,' and a friend says: 'And I will take twelve of the same buttons, please,' do your two purchases look alike? Would the two lots weigh the same? Would the owner of one of the lots be any the worse or the better off, if the parcels were changed by accident? If the twelve buttons were sewn on to your coat, would any one know that they were not the dozen? If we wanted to play a game with twelve counters and had no proper counters, we might use the twelve buttons for counters; would the dozen do instead? Yes, just as well. The dozen is twelve, and twelve is a dozen; and for every purpose for

which one could be used, the other would do just as well.

You say twelve pence are a shilling. Do they look like a shilling? Are they the same colour, size, weight? If I wanted things to use instead of counters, I might use twelve pennies: would a shilling do instead? No. Sometimes in cooking, if we have not small weights, we use a coin as a weight; we might be told in a cookery book to take the weight of a sixpence or shilling of carbonate of soda or of some spice. How would it be if we used twelve pence instead?

Twelve pence are not a shilling, not in any way like a shilling. Why do you say they are a shilling? They are of the same value. Value for what? Not for counters, nor for weighing things.

Twelve pence are of the same value as a shilling when we want to buy things. Yes, now we have it right; twelve pence are not a shilling, and cannot be used instead of a shilling for any real use of the things themselves. But the chief purpose of coins is to exchange for other things; and, *for that purpose*, twelve pence are of the same value as a shilling.

Then if that is what we mean, why don't we say so? Why do we say 'Twelve pence are a shilling,' when we don't mean it?

Ah! why? Now we have come to something which it is very important you children should understand and remember.

Think how often, in your arithmetic, you have to make the change from pence to a shilling or from a shilling to pence. How would it be if the teacher had to stop every time and say, 'It is arranged by law that, for purposes of buying and selling, twelve pence shall be considered as of the same value as one shilling'? The teacher would be tired of saying it; you would be tired of hearing it; it would waste time, and disturb you from attending to your lessons; so it has been arranged that people may always say, 'Twelve pence are a shilling,' for shortness, whenever the business on hand is using coins as money, or talking of them as money, as things to buy other things with; but not when any other business is on hand.

'Twelve buttons are a dozen' is true always, everywhere where people are speaking English, whatever use they mean to make of the buttons. But 'twelve pence are a shilling' is not really true in itself; it is true as far as the business on hand is concerned, whenever people are talking about using coins to buy things with.

You might go some day to a class in what is called physical science, where the business on hand is learning about the weights of things

or the qualities of things; or to a class in cookery. If the teacher told you to go and fetch a dozen bottles, and you came back and said, 'Here are the twelve bottles you sent me for,' nothing would be said about it; you would be all right. But if the teacher told you to take the weight of a shilling in some powder, or to take as much as would lie on a shilling, or to see what effect some liquid had when it touched a shilling, and he found you using twelve pennies instead, there would be something said to you then! You would perhaps be told that you never could learn anything while you were so stupid and clumsy and inaccurate.

I don't suppose you would ever make just that particular mistake. But I have known many children hopelessly puzzled over sums and other things; and, when I came to see what puzzled them, I found they had been taking something some teacher had said, as meant to be true in itself, when it was only a shorthand sort of sentence which meant something else: a shorthand which was true for one purpose but not for another.

Try to understand always whether your teacher means what he says to be true always and everywhere; or whether he means it as a bit of shorthand talk fit for that particular

class. If you cannot find out for yourself, ask. Never go on using a sentence till you are sure whether it is meant for literal truth or convenient shorthand. Think of the 1, the stroke which the old people cut on their tallies, and which meant, not one thing, but once putting all the fingers up and then down so as to begin counting again. Arithmetic is full, from beginning to end, of just such nice, clever, convenient bits of shorthand as that. Sums are difficult and puzzling chiefly because children forget this.

V

KEEPING ACCOUNTS

Suppose you have a shilling and spend three-pence, what have you left? Ninepence. Is that all that is left?

Suppose that you go out with a shilling in your purse and spend three-pence on buying flower-roots, what do you bring back? Nine-pence in your purse. Anything else? Anything not in the purse? The roots. Well, is that an end of the matter? What is to happen next? The roots have to be planted, and then watered; or else they will die, and you will have wasted your money.

You might have bought something else, not roots. But whatever you did with part of your money, something of some sort is left besides the change. If you bought food, it would be there ; and must be seen to that it might not be wasted. If you ate the food while you were out, you would bring back the strength you got out of it ; or if it was unwholesome food, you might bring home ninepence and some very uncomfortable feelings. Whenever money has been spent, something remains besides the mere balance of cash.

Yet if you have been taught to keep accounts, or if any grown-up person allows you to see how she keeps accounts, you know that when the money spent one week has been subtracted from the cash in hand, the page on which the things bought during the week have been written down is turned over, and only the remaining cash is entered on the new page for next week.

Now shut your eyes for a minute or two, and think. See an account-book open. On one page is written :

'Mother gave me 1*s.*'

On the other page is written :

'Plants, 3*d.*'

You subtract the 3*d.* from the shilling, and write 9*d.* Then you turn over the leaf, and

on the next page you write, ‘ Brought over : 9*d.*’ The writing about the plants is hidden out of sight ; nothing is on the open page except the account of the money still left in your purse. The roots are not in sight ; they are in another room, waiting till you can go to them. But they have not gone out of the world : they are there, waiting ; and you will have to see to them presently.

Then why is nothing written about them in the new page of the account-book ? Think about that a little, and we will have a lesson about it next time.

VI

DIS-MEMBERING AND RE-COLLECTING

THE question you were to think over was : Why, when we turn over a page in an account-book, we enter on the next page nothing that was on the last page, except the balance in cash ; we make no mention of the things bought last week, though those things may still be left for us to deal with. The plants we bought last week still need watering ; part of the food we bought may be there still to eat ; the strength we got out of what we ate may remain to be used ; the illness we got if the food was

unwholesome may be still uncured. Yet we make no mention of these things in this week's accounts.

Why not?

I think you feel why not, though perhaps you cannot quite explain it. To-day we are going to have a little talk about that question.

Human beings, you and I for instance, are finite creatures; that is to say, we cannot be everywhere at once or do many things at once. Our bodies are made so that we can see only a short way across our big world; and our minds are made so that we can only attend to a small part of any big business at once. All the rules of Arithmetic are made to help us to do big sums by attending to little bits at a time. If our minds were bigger we could do big sums, straight off, without any rules; but, as it is, we have to attend to a bit at a time; and the rules are made in order to fit the bits together properly. It is not only sums that have to be attended to a bit at a time, in proper order according to rule. Suppose you are going to a school-treat or picnic. If you were fairies, you could start off straight away; and the proper clothes and boots would grow on you, if you needed them, as you flew along. But you are not fairies, so the clothes and boots have to be put on first, and carefully too, or

you cannot go. Suppose you were thinking about what you would do out of doors, and not tying up your boot-laces properly, mother might say: 'Come, attend to your boot-laces now; and when they are tied up safely, you can think of out of doors.' But 'out of doors' has not gone away because you have to put it out of your mind for a little while and attend to your boot-laces. It is waiting, waiting, till you are at leisure to attend to it. So it is always. When we put anything out of sight and out of mind, in order to attend to something which for the moment we must attend to, the thing we put out of mind is not gone out of the world; in some shape or form it is waiting, waiting, waiting, and will have to be reckoned with some time or other.

Now we will do a little sum.

How many days has a man lived who has lived forty-seven years?

If I asked you how many days a child has lived who has lived two weeks, you could tell me straight off, without any need to write on the black-board. There are how many days in a week? (7) and twice 7 are? (14). If your minds were giant minds, you could answer the other question, about the days in forty-seven years, just as easily as that one. But we, you and I, have such tiny, helpless little minds that

we cannot manage a question like that. If we were giants we could pick pears off the top of a very big tree; but human people have to climb up on ladders to reach high-growing pears. And in the same way we have to make a sort of mind-ladder before we can reach such big numbers as the days of forty-seven years. What are we going to do? Write on the board. Yes; tell me what I am to write.

A **3** and a **6** and a **5.** And then, underneath, a **4** and a **7.**

What does the 3 stand for? Three hundreds of days. And the 6? Six tens of days. The 5? Five single days. The 4? Four tens of years. And the 7? Seven years.

Now tell me what I am to write. Seven times five are thirty-five . . . Why, what are you about? I asked you to find the number of days in forty-seven whole years; and here you are, telling me how many days seven times five days are.

Had you *forgotten* the hundreds of days and the tens of days and the forty years? Yes, for the moment; we had to forget them, to push them out of our minds, so that we could attend properly to just the bit we were doing at a time. Go on then, seven times five are **35**. There, I have written that down. Have we done now? Is the sum right? Will you go home and tell

your mothers that you have found out at school how many days there are in forty-seven years, and there are just thirty-five days in all those years? Much good there would be in going to a school where that sort of thing was allowed! Thirty-five is only the first rung of the ladder; we haven't reached our answer yet. The three hundreds of days and the six tens of days and the forty years are waiting quietly, till we have finished attending to the seven times five; next it will be the turn of seven times six tens to be attended to, while the hundreds are waiting, waiting. And now we multiply the three hundreds by seven; and all this time the forty years are waiting, waiting, waiting, still.

Well, now we have attended to each bit, and here are all our bits written down, one under the other[1]. But which of all the bits is the answer? Which is the true right answer to the question we began about?

Alice, you say, $7 \times 6 = 42$. And, Mary, you think that $4 \times 5 = 20$. And, Julia, your opinion is that $4 \times 6 = 24$.

Which is the true view of the matter? Are you going to quarrel about which is right? Or would you rather have a nice long solemn argument, each trying to prove that she has the true, the only true, answer to the question I asked?

[1] See note at end of chapter.

You won't do either? Well then, what will you do? We must fit them all together to see what they all come to, before we have a right to tell people how many days there are in forty-seven years; because our minds are so tiny, and the number of the days of the years are so very, very big. We have to make ourselves forget some things while we attend to other things; but before we dare tell any one that we have found the actual truth, we must call back all that we made ourselves forget, and try to re-collect.

Well then, 2 and 4 are 6; and 1 are seven. Is that right? No? Why not? Some of the figures mean hundreds and some tens and some only single days, and we must sort them properly before we fit them together, and arrange them so that each sort of figure shall stand for its true value.

Now we understand better the question about the account-book. The use of the book is to keep account of money. Our minds are so small that we cannot think of cash accounts and other things at the same time. We wrote down once: 'Plants, 3*d.*,' because we may some day wish to look back and see what different things cost us; but once writing it was enough: we do not wish to be constantly reminded of plants and other nice things, while we are busy adding up

accounts; we try to forget the things and attend to the cash accounts, just while we are using that book. But when we have done the accounts and shut up the books, we should recollect the things; for everything that we have spent money on is still somewhere, in some shape or form, waiting, waiting, waiting.

And when we go to recollect the things, we must try to arrange them in their proper order so as to give to each its true value and meaning.

That is enough for to-day. Sit slack, shut your eyes, and rest before you go to the next class. I am going to give you two mind-pictures:—

A little boy was so excited about going to a treat that he would keep talking about it, and would not tie his shoe-laces properly. On his way downstairs his lace came untied; he stepped on it and fell and sprained his ankle, so he could not go to the treat after all.

A servant was asked to get some children dressed to go out. She got them tidy, and their boots nicely cleaned and tied on; the children felt impatient because they were in a hurry to go out; but they knew they could not go till they were dressed, so they were good and quiet. When they were ready, they thought they were to go. But the nurse grumbled and said:

'Now you are dressed and all tidy and clean, I don't want you to go out, for fear you should tumble your hair or get your boots dirty. I was told to dress you, and I have dressed you: that was the important thing; that was my real duty. Going out is all nonsense. Sit still all the afternoon, and look at your nice clean boots.'

Try to fancy how those children would feel, and what they would think. Try for a minute or two to fancy yourselves in the place of those children, and to think what you would feel like.

Now open eyes and stand up. Go and get ready for the History Class.

NOTE.

The answers had better be written on the board at first in this form:—

7×5	=	35.
7×6	=	42.
7×3	=	21.
4×5	=	20.
4×6	=	24.
4×3	=	12.

VII

WEIGHTS AND MEASURES

On the Table:—A set of weights, one-ounce, two-ounce, four-ounce, half-a-pound, one-pound.

WHAT is this? A weight. What weight?

An ounce. What is it for? To weigh things with. Yes, this ounce weight is one of the things that has no proper use of its own; its use is to make other things more useful by keeping them exact and orderly.

Let us think of the uses of weights. They are used in shops, for the shopman to know exactly how much sugar or tea or meat he is to give for his customer's money.

But they have another use as well.

Very long ago people had no notion how to make cakes or puddings. When they caught an animal, they roasted it by a fire; and when they found roots fit to eat, they roasted them in hot ashes; and that was all they knew about cooking. Then they found out how to make pots of some kind, and then they found they could boil things in water. Gradually they found that some things improve the taste of other things, when they are boiled together: salt improves soup, and sugar improves puddings. But there must not be too much of one sort of flavouring. People began to try to find out how much of each kind of thing made the nicest mixture. They could find out only by trying. That is what we call experiment: a woman finds the soup too salt or not salt enough one day, she thinks it nicer when she puts in less salt or more salt; then

she tries to remember for the future what is the right quantity.

But it is a pity for each one to have to make all the experiments for herself. It saves time and trouble, if one person's experiments can be made useful to other people. Suppose a woman is a very clever cook and finds out how to make things just right, it is well for her to be able to tell other women just what she found answered best. The other women's husbands and children may not have quite the same taste as the first woman's; they may like a little more salt or sugar or spice than the first woman said; but still, it is useful to know what other people find answer, and then we can alter it to suit ourselves and our families.

Now how is the woman, who has made a nice cake, to tell others how much she put in? She has to use weights or else measures. She says: 'I took a pound of flour, and a quarter of a pound of sugar, and half a tea-spoonful of baking powder,' and so on; weights and measures are a language in which one person can tell others how much of anything to use.

Well, a woman read in a book a receipt, as it is called, for making a nice small cake. One of the ingredients was an ounce of lard. But she was preparing for a tea for a great many people, and she wanted to mix all her dough at once

for several cakes. She wanted to mix sixteen times as much dough as the receipt told about. So she had to use more lard, sixteen times as much as the receipt told about. She had only one ounce weight. Look at my set of weights here; there is no other of this size. Most sets of weights have only one of each size. The woman I am telling you of had only one weight of this size, and she wanted to weigh sixteen times as much. What did she do? Did she weigh out each ounce separately? If you think, you will see she could do something handier and quicker than that: she could weigh the whole sixteen at once with the pound weight.

Now I should like you to go over the weights and see how they are arranged for use.

If we want to weigh one ounce we use? The ounce weight. If we want to weigh two ounces? The two-ounce weight.

If we want to weigh three ounces, what shall we do? There is no three-ounce weight. Put in the scales the two-ounce and the one-ounce.

For four ounces? The quarter of a pound weight.

For five ounces? The quarter of a pound and the ounce.

For six ounces? The quarter of a pound and the two-ounce.

For seven ounces? The quarter and the two-ounce and one-ounce.

For eight ounces? The half-pound.

(The teacher goes on thus up to thirty-one ounces.)

So, you see, we can weigh any number of ounces we please, up to thirty-one ounces, by having only these five weights, one ounce, two ounces, quarter of a pound, half a pound, and one pound.

If we want to weigh more than thirty-one ounces, we shall have to get other weights; such as two pounds, four pounds, &c.

But we can weigh thirty-one different quantities with only five different weights.

VIII

MULTIPLYING BY MINUS

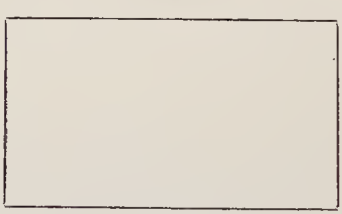
Many persons who have learned a certain amount of Algebra are confused between what are called negative quantities, (movements, or actions), and the mental act of negation of quantity, (movement, or action); i. e. between the sign — which indicates active fact in the direction contrary to the one (arbitrarily) chosen as positive, and the symbol 0, which states denial of existence of fact. Practice should be given in exercises which involve these signs and show the distinction between them.

It is also desirable, for many reasons, that children should be accustomed to use letters for unknown

quantities at an early age. It assists the imagination in keeping a clear distinction between a quantity in itself and the effect on the quantity of a certain operation or group of operations. Whatever a number may be, the group of operations represented by $-7+4-9+2$ diminishes it by 10. The statement $x-7+4-9+2=x-10$ has a meaning which can be made intelligible at an early age; whereas the mere statement $-7+4-9+2=-10$ is confusing to a child's imagination. Even for those who are never to learn formal Algebra, the practice of using letters, for numbers which are to be the subjects of operation, imparts to Arithmetic something of Algebraic clearness of thought, which will be of use in itself; while those who are to learn Algebra should not be exposed to the mental violence of being introduced to Algebraic notation for the first time, just when they will have to begin to learn the actual manipulation of unknown quantities by processes properly Algebraic. The following is a specimen of the kind of form in which elementary exercises in ordinary sums may with advantage be given.

BLACK-BOARD.

T



P

THE figure on the board is supposed to be a counter in a shop. We are going to do a sum about buying something over the counter. What

does the shopman call the thing in which he keeps his money? A till. Well then, we will call the amount of money in it T. What do we call the thing a customer carries money about in? A purse. We will call the money in this customer's purse P. We do not know how much T is, nor how much P is; and we do not want to know just now. Provided that the customer has brought enough to pay for what she buys and the shopman has plenty of change, that is all we shall need to know at present. When we do not know a number, we often call it by the name of some letter.

The customer sees boxes of sweets marked 11*d.* She takes a box and pushes a shilling towards the shopman. What does he do? Does he push anything towards her? A penny.

Does the shilling which she gives to him add to what is in the till or does it make it less? Adds to it. When we add two things together, we put this mark between them +. So now we write what is in the till **T + 1***s.* T means what was in the till before the customer came in. T has not altered. What is now in the till is **T + 1***s.*, which we read: T plus **1***s.* But what has the shopman to give to the customer? **1***d.* Is that added to **T + 1***s.*, or taken from it? Taken from it. We write that this way: **−1***d.*, and we call it minus **1***d.*

So now this is how we write what is in the till: **T + 1***s.* – **1***d.*

Just for to-day we are thinking about things as the shopman thinks about them; we are agreeing with him. It is his till to which a shilling has been added, and his till from which a penny was taken. He thinks about the shilling as plus and the penny as minus. Some other day we will see what the customer thinks, but the shopman's thinkings are enough for to-day. He thinks pushing something from P to T is a plus action; and pushing from T towards P is a minus action. As long as we are thinking like the shopman, everything that goes from P towards T is marked plus; and everything that goes from T towards P is marked minus.

For to-day, P to T is the plus direction. T to P is the minus direction.

By such a lesson as the above, we lay a skeleton, round which may be grouped a great variety of examples in ordinary addition and subtraction of money. The customer may ask for several articles in succession, and a bill be made out. Several customers may come in, and require amounts of change. All the money which passes should at first be registered as added to or taken from T. Sometimes the customer has the exact coins needed. In such case no change will pass; this fact should be carefully recorded by writing – 0.

When the teacher wishes to extend the scope of the operations, he should go back to the one shilling given and one penny change; so as to leave the children's

thoughts free to understand the operation involved, undistracted by questions of number.

We are going now to look at the business of paying for things, from the customer's point of view, to see how she thinks about it. When she pushes a shilling towards the shopman, is it added to or taken from P? Taken from P. And the penny of change is? Added to P. Then the customer's account now is? **P − 1***s.* **+ 1***d.*; The shopman's is? **T + 1***s.* **− 1***d.*

After this, money sums should be done sometimes from the shopman's, sometimes from the customer's point of view. Occasionally, a sum should be written out from both points of view. Every shop transaction should be dealt with either as an addition to or a diminution of T or P; or, preferably, as both.

At a still later stage we suggest:—

We are going to see whether the money has been paid and the change given all right. For that purpose we must know what was in the till before the customer came in. T was £3 9*s.* 6*d.* So on the shopman's side we ought to have **£3 9***s.* **6***d.* **+ 1***s.* **− 1***d.* P was £4 7*s.* 8*d.* Therefore the customer ought now to have **£4 7***s.* **8***d.* **− 1***s.* **+ 1***d.*

Then more sums are grouped round that conception. There is a further stage of the same scheme.

A customer asks for twelve articles value 19*s.* each. She lays down 12 sovereigns. The

shopman hands back a shilling change for each sovereign : **12***s.*

One of the articles is then found to be faulty. There are no more in stock. The customer says: 'Never mind; eleven will do for the present.' The shopman hands her back £1. Who has too much now? Who must give a 1*s.* in change? The customer.

The account then stands thus :—

On the shopman's side,

$$\mathbf{T} + £\mathbf{12} - \mathbf{12}s. - £\mathbf{1} + \mathbf{1}s.;$$

and on the customer's side,

$$\mathbf{P} - £\mathbf{12} + \mathbf{12}s. + £\mathbf{1} - \mathbf{1}s.$$

In order to emphasize this lesson, it should also be written out in another form :—

$$\begin{array}{r} £\mathbf{1} - \mathbf{1}s. \\ \times\,(\mathbf{1\ doz.} - \mathbf{1}) \\ \hline £\mathbf{12} - \mathbf{12}s. \\ - £\mathbf{1} \quad\quad \end{array}$$

Now then, what does $-\mathbf{1} \times -\mathbf{1}$ come to? In which direction must the last shilling be pushed? From P towards T. But that is, from the shopman's point of view, the plus direction. So minus multiplied by minus comes to plus.

Now from the customer's point of view :—

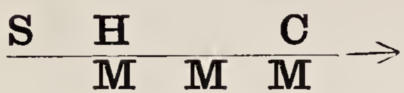
$$\begin{array}{r} - £\mathbf{1} + \mathbf{1}s. \\ \times\,(\mathbf{1\ doz.} - \mathbf{1}) \\ \hline - £\mathbf{12} + \mathbf{12}s. \\ + £\mathbf{1} - \mathbf{1}s. \end{array}$$

This lesson carefully gone over, with the simple numbers given above, and afterwards repeated with more complex sums, would save all future trouble about the much-vexed problem of 'minus multiplied by minus.'

IX

ADDING MINUS

Black-board.



S is a shop where a boy works. H is the house where he lives and sleeps. C is a house where a customer of his master lives. Each M is a mile-stone on the road. The distances from **S** to **H** and from **M** to **M** are, each, a small picture of a mile along the road.

Now, do you remember what those letters stand for? Because I don't want you to have to think about that presently when I am explaining something else. So say those letters over: S, shop; H, boy's home; C, customer's house; M, mile-stone. Again: S, shop; H, boy's home; C, customer's house; M, mile-stone.

You see which way the arrow points. Do you remember that we had a lesson about

money pushed across a counter? When it was pushed across one way, it added to the shopman's money; so, when we were doing his accounts, we called that direction plus; and when he pushed change back, we called that minus. Miles on a road are sometimes counted in the same way: we call one direction plus and the other minus. Well, the boy who works at the shop lives one mile from the shop in the direction that we are going to call plus; what then shall we call a mile in the opposite direction? A mile *minus*; he walks a mile *minus* in going to work; every evening he walks a mile *plus* in going home. C is two miles further on than H. So all the distance from S to C is how many miles? In what direction? S to C is three miles *plus*.

One night the master says: 'Bill, there is a parcel to be taken to Mr. Smith's house (C). You live in that direction; you must take it.'

What then has Bill got to do? How many miles plus has he to walk? Three; and what has he to do next? He must get home somehow. What must he do? Walk two miles minus. So three miles plus and two miles minus brings him to the same place, H, as he gets to on other evenings by walking one mile plus.

We said that the boy's task, set him by his master, was to walk three miles plus. He

neutralized or counteracted part of that walk by adding to it a walk of two miles minus.

But now suppose that before Bill starts with his parcel a workman says to the master: 'I am going to see a friend who lives close to Mr. Smith's house; may I take the parcel, sir?' And suppose the master gives leave. What has the workman done to Bill's task of three miles plus? He has *taken off* from it two miles plus. Bill still has to walk one mile plus, because he must get home.

If the workman takes the parcel, he *subtracts* or takes off from Bill's task, two miles plus; if Bill takes the parcel he neutralizes two of the plus miles by *adding* to his three miles plus another walk of two miles minus. His walk in either case ends at the same point H. But now I want to know:—are the two ways of getting home the same in any other respect? For instance, will he get home to supper at the same time? No, he will be more than an hour later for supper if he takes the parcel himself. And what about exercise and health and all that? If Bill is cashier, and has been sitting still all day in a close shop, the five miles walk, three plus and two minus, would do him a great deal of good. But if Bill is a delicate little chap and has been on his feet all day, that five mile walk instead of one mile might make him

quite ill. It would make a difference to wear and tear of shoe-leather, and to the length of time he would have for playing with his sister before going to bed. The only thing about which adding two miles minus *comes to the same* as subtracting two miles plus, is the point, one mile from the shop, where he gets to at the end of his walk.

So you see, if you are asked a question about the point which Bill will reach at last, it will be quite right to say, 'Adding minus two comes to the same as subtracting plus two.' If you are asked about Bill's health, or his shoes, or the time he gets his supper, you must not say anything of the kind; because, in relation to those questions, it would not be true.

Mind-Picture.

Bill trudging home tired; near the middle M; with his back to C and his face towards H. He is looking tired, and is glad to have got as far as M on his way back.

X

DIVIDING AND SHARING

What is the half of five? Two and a half. If two children are to share five ounces of anything, what is the share of each? Two and

a half ounces. Is that so always? Are you sure? Well now, let us see.

Suppose two children are to share a cake weighing five ounces, the share of each is? Two and a half ounces of cake. How do we divide it? Do we give one all the top and the other all the bottom? No, that would not be a fair division; we cut it so that each gets half the currants, half the sugar on the top, and half the bit of candied peel in the middle, as well as half the dough.

Suppose five marbles, weighing an ounce each, are given to two boys to play with, what is the share of each? Two marbles; and there will be one *over* (as you say in division sums). What would the boys do with the one over? I think they would put it on the ground between them, and each shoot at it in turns, with the marbles that were his own share.

Suppose I have a doll weighing five ounces, and give it to two little girls. What is the share of each? Will they cut it in two; and each have half a dolly? Shall each have one leg and one arm and half a head; half the calico, half the plaster, and half the saw-dust? No. Well, what would be the share of each? If it was a naked doll, there would be its clothes to make; they would do the work between them. When they had time to nurse it for an

hour, they would each nurse it for half an hour. When they had it to tea, they would sit one each side of it. That would be the fair sharing of a five-ounce dolly.

Now suppose a flower-bulb, weighing five ounces, is given to two children, how would they share it? Would they cut it in two down the middle? No; they would plant it, and take turns at watering and tending it.

Children who have become mechanicalized, here add: 'And share the flowers when it blossomed.'

Do you mean what you are saying? Just think a minute. Babies might cut a hyacinth up for the sake of sharing the flower-bells between them, and each being able to say, 'These are *mine*'; but I never saw two children old enough to tend plants who would do such a stupid thing. What would you really do if the plant were yours? Keep it growing, to look at, and show to friends; and how about the sharing? If one child looks at it, does that prevent the other seeing it? If one smells it, does that prevent the other smelling it? If one child brings in her special school-friend to see it, does that prevent the other from bringing in *her* special friend to see it too? No. But neither child can say: 'This is *my* plant.'

So now this is what we have come to.

If two people are to share five ounces of

cake, each is to have two and a half ounces of cake, each share to contain half of each kind of ingredients that the cake is made of.

If they are to share five marbles weighing an ounce each, the share of each is two marbles and the fun of shooting them at the fifth marble.

If they are to share a five-ounce doll, the share of each is half the work of dressing it, the fun of nursing it for half the *time* they have to spare; and the fact of sitting beside it at tea just as if it were all her own.

If they are to share a flower-bulb, the share of each is, half the work of tending it, all the pleasure of seeing it, all the pleasure of smelling it, all the pleasure of letting friends see it; and none at all of the pleasure of saying: 'This is mine, my very own, and no one may look at it or smell it without my leave.'

XI

IN WHAT CONSISTS ECONOMY

Black-board.

A.	B.
Things such that no one else can have the part thereof that I have.	**Things such that other people can have the part thereof that I have.**

You remember what we said about sharing cake and sharing a lily-bulb. If two children share five ounces of cake equally, the share of each is? Two and a half ounces. If one has three ounces, the other can only have? Two. If one has four ounces, the other can only have? One ounce. If one has five ounces, the other gets? None. Now look at the black-board. On which side shall we write **cake?** Under A.

Now about the lily-bulb. If two children share a bulb, the share of each is?—

Half the labour of growing it.

All the pleasure of seeing it grow.

All the pleasure of smelling the flowers when they come out.

All the pleasure of seeing friends enjoy the flowers.

None at all of the pleasure of saying: 'This is my very own, and no one else can share it.' So we shall write the **bulb** under? B.

Suppose two people grow bulbs for sale, they get? Money. What happens to the money? It is shared between them. If they get six-pence for a bulb, the share of each is? Three-pence. If one has four-pence, the other can only get? Two-pence. We will write **money got by selling bulbs** under? A.

Now I want you to think about a little row

of houses, six comfortable houses, all about the same size, built near together on a common, away from shops. The gardens are as yet scarcely planted at all. Suppose one of you is the mistress of the end house of the row.

One morning, two hawkers come up the road and come past your house first. One has on his barrow a lot of nice fresh fruit, about as much as he thinks the people in your row will want to buy. The other has young plants.

Now remember, we are not here talking of shipwrecked sailors, or desert islands. I am not asking what people should do who are shut up together, with not enough food. I am not asking what is the heroic thing to do in exceptional circumstances; but of what is the common-sense thing to do in commonplace circumstances. There is enough fruit on the barrow for every one in the row; and every one has enough money to buy some. We will admit at starting that the first duty of a housekeeper, in ordinary civilized society, is to see that there is proper food in the house for all the household; so the first thing you do, when the hawkers call, is to buy as much fruit as your household will need while it keeps good.

The people from one of the other houses are coming in to tea; so you buy enough for them to have plenty as well as your own family.

We will suppose that all the rest of the necessary provisions are in the house already. You have just bought fruit enough; and you have a few shillings which you decide to spend on 'extras.' There are two things you can do with them.

It comes into your head that it would look rather grand if your neighbour were to see on the table a huge heap of strawberries:—two or three times as many as could be eaten that day. Will you buy that extra fruit? Or will you buy some plants for the garden?

Let us see what would come of each of these proceedings.

Suppose you spend your spare shillings on buying more fruit than you need, some other family will not be able to get any. So we must enter **strawberries** on the A side of the black-board.

If you buy up that fruit, what happens to it? A good deal is left for next day; not quite as nice as it was the first day. No one can enjoy it now quite as much as the people who got none would have done if they had got it while it was fresh. Everybody in the house is perhaps tempted to eat a little more than they otherwise would do, because it is there and must not be wasted. Yet in spite of that some of it is wasted; goes really bad

and has to be destroyed at last. Meantime there has been about the house a faint smell of not quite fresh fruit, which takes away everybody's appetite and makes every one uncomfortable. For the next few days, no one cares as much about fruit as they did before.

You will often find this the case about things in the A class of this division. (Point to blackboard.) It is right that you should have *enough* of each of them; but the least bit more than enough is too much, too much even for your own good.

Well now, suppose that instead of buying too much fruit you buy plants for your garden, what happens? All the people in the house enjoy them; visitors enjoy them; and the people who pass by on the road enjoy them too.

We will write the **plants** down in Column B.

I have known many servants who liked living in houses where the mistress bought what is called in Ireland 'lashings and leavings' of all sorts of food-things; they called a mistress stingy who bought only as much as was of real use. You might see, in such houses, ends of joints going bad, bits of loaves on the floor among dirty boots, milk going sour, fish going stale, candles guttering away in waste;

and the garden quite neglected because the family were too poor to keep it up and had no money to spend on plants. And the servants thought such masters 'rale gintry.' You all know that that was because these servants were ignorant and uneducated. I think you will find that this is a good test of true education.

(The teacher now rubs out the instances, leaving only the headings of the two columns; and writes under them this sentence :—)

True education tends to make people satisfied with just enough of the things in Column A, and leads them to spend spare time, money, and energy on things in Column B.

Read what is on the black-board. Copy it in your books. Read it out. Shut your eyes and try to say it. Learn it by heart (in preparation time).

Now I am going to tell you something rather difficult. You will not quite understand it yet, but if you think about it and fix it in your minds you will understand it some day, and it will help you to understand many other things.

You might, if you could afford it, buy more kinds of plants than you could attend to

properly, or than the garden could well hold. Some people do that kind of thing; they cannot make up their minds at once to do without what they cannot use; so they crowd up their houses and premises with things that are in the way.

Shut your eyes and think of this for a minute or two:—Plants that are crowding each other up, and preventing each other growing, or that are not planted and are littering up the house, are not in Class B; I am not sure that they get as far as to be properly in Class A.

XII

ECONOMY OF MIND-FORCE

True education tends to make people satisfied with just enough of the things in Column A, and leads them to spend spare time, money, and energy on things in Column B.

Black-board.

A.	B.
Things such that no one else can have the part that I have.	**Things such that other people can have the part that I have.**

You know that when I want to make you remember a sentence which I have written on the board, I let you all say it at once. But when I want to help you to understand something, or to see whether you understand, I allow only one child to answer at a time. I will write:—**repeating a thing to help me to remember**, in Column? B.

I will write:—**showing how much I know and how far I understand**, in Column? A.

There are several things you have to learn in school, besides the actual lessons in the books, such as sums and geography. One of the most important is to answer questions properly. You must learn to think what was the exact question asked; to think what the question means; to answer it quickly, quietly, and politely; to tell the truth, the whole truth so far as you know it, and nothing but the truth, about that question; to put it in plain words, and not to use unnecessary and roundabout phrases. All that needs practice. It is the teacher's duty to give each child a share of such practice—just as it is the housewife's business to give every child a proper amount of food. And as we said, while Mary is answering, Claire cannot answer. Therefore we put:—**practice in answering questions**, in Column? A.

Another thing that is necessary to learn is to listen to other people's conversation, without interrupting till your turn comes. If Alice is answering me, and Mary is sitting quiet, can Claire sit quiet too? Yes. Then we will write :—**practice in listening to conversation without interrupting till my turn comes**, in Column? B.

But we must not sit idle while other people are discussing interesting or useful things, we must learn to profit by what is said. If I am teaching Alice and making her answer, and Mary is listening and learning by what Alice and I say, can Claire listen and learn too? Yes. Then we will write :—**learning from what is going on**, in Column? B.

It is necessary to collect a good stock of mind-pictures for our future use. While one child is sitting nice and quiet making a mind-picture, does that hinder the others from doing so too? No; the quieter each one sits, and the more steadily she makes her mind-picture, the better every one else can do the same. We will write :—**making mind-pictures**, in Column? B.

It is necessary that a child should not pass up into a higher class (form, or standard) till he is quite fit for it. Therefore, there must be pass-examinations, to see who is fit. If one child passes an examination creditably, does

that hinder another from doing so? No. We will write:—**passing school examinations creditably**, in Column? B.

It is necessary that Governments should know who is fit to become a doctor, or a lawyer, or a school-teacher, or a Civil Service clerk, or a postman, because they ought not to allow unfit persons to undertake responsible posts. If one man shows he knows his work properly, does that hinder another from showing that he knows it too? No. We will write:—**passing professional pass-examinations**, under Column? B.

But for some purposes it is right that the very fittest persons who can be got should be chosen for a post. It is sometimes right therefore that the Government should know, not only who is fit, but who is most fit. If one man is at the top, can another be at the top? No. We will write:—**being at the top**, in Column? A.

Now let us read over carefully our two columns.

(Read out the columns.)

Now I am going to speak about something which puzzles many children. You are always being told that it is kind to help other people and share with them, yet if you help another child at an examination you are told it is naughty; and if you let another child help you

you are punished. Even at class, if I ask one child a question, and another whispers the answer to her, or writes it for her to see, I stop you at once.

Let us see what all that means.

If I say to one child, 'I do not know whether any dogwood grows in this neighbourhood; will you keep a look-out when you are out walking?' Would it be right for another child to help her to look for it? Yes; because the thing I want to know is whether there is dogwood; and the thing the child wants to do for me is to find the dogwood if it is there. Two pair of eyes are better than one; two children, by helping each other, are more likely to succeed in doing what is wanted than one alone.

But if I ask one of you a question at class—for instance, what are nine times eight?—I do not want to know what nine times eight are; I know that already. What I want is, first, to know whether that child remembers nine times eight and can say it without help; and, secondly, to give that child practice in grasping for herself the question she was asked and answering it for herself. If another child interferes or tries to help, she does not help, she hinders the very purpose for which the question was asked.

So it is at examinations. The examiner does not want to know the answer to the questions

he has asked; he wants to know which children are fit to begin the work of a higher class. If you 'help' a friend (as you call it) to answer questions, you are really hindering her progress, by helping to get her into a class she is not fit for. You help her to be set to lessons too difficult for her. You help her to have duties which she cannot properly do. You help her to brain-muddle and overstrain and many sorts of bad things.

BLACK-BOARD.

When the object of doing a thing is to get the thing done, it is right to accept the help of other people.

When the object of doing a thing is either to get practice in doing it, or to show that I can do it, I must not accept help from other people.

There is something that I should like to tell the elder children just here. Little ones must please sit still, and practise trying to learn what they can from conversation between their elders.

There are, in every subject, some parts which *can* be learned on purpose to show that you know them; for instance, on purpose to pass examinations in them. There are other parts which *cannot* be learned so, which must be

learned while you are *not* thinking of showing what you know. Now you can never really thoroughly understand either of these parts unless you also learn something about the other. You want to know why this is so? Well, I will answer that question after you have answered me a few questions. Which makes a man strongest for exercise, walking all day, or leaving off to take food sometimes? Which has the better nourished body, the boy who sits trying to stuff down food all day, or the boy who does a fair amount of running about between times? Which will put forth most leaves and fruit to show, the plant that has been cut from its root and now has only branches, or the plant that is growing from a root hidden under ground?

I once knew a selfish, greedy little boy, who would rather make himself ill than let the servant (who made the pudding, and brought it to him, and was going to wash up his plate) have any herself. Do you think that boy could learn the true use of food, and the proper care of his digestion, while he thought of his food in that sort of way? You know that he could not learn so. Why? You cannot explain, but you know he could not.

Well, tell me why all those things are so, and then I will tell you why you understand your

examination work best on the whole, if you do some portions of work which do not tell at examinations at all. It is our 'nature to,' as the old rhyme says. We are made so, and we cannot help ourselves.

All the great men whose names live for centuries after their death, did a great deal of work which the world never heard of. Men who will do nothing except that part which they can show, never do anything which lasts.

A minute or two of silence.

Now all of you read aloud what is on the black-board. Copy it into your books. Read it again. Shut your eyes. Try if you can say it. Learn it by heart (in preparation), and say it to me next time.

I am not going to give you a mind-picture to-day. Shut your eyes and sit at ease, and try to put together in your minds the different things we have been talking about.

Five minutes' silence.

XIII

EXERCISE IN RELEVANCE; INTRODUCING IDEA OF PROBABILITY

Suppose a pony is shut up in a field alone. How many heads will be in the field? How

many legs? Eyes? Hoofs? Tails? Hands? How many pieces of mischief do you think will be done in an hour? Suppose another pony is turned in. (Repeat same questions.) Twenty ponies come in. (Repeat.) A hen comes in. (Repeat.) A boy comes in. (Repeat.) Another boy comes in, the second boy has a monkey. The master comes in. (Repeat the questions each time.) Now let us put all that on the black-board.

	Heads	Legs	Eyes	Hoofs	Tails	Hands	Probable Bits of M.
1st Pony	**1**	**4**	**2**	**4**	**1**	**0**	
2nd Pony	**1**	**4**	**2**	**4**	**1**	**0**	
20 Ponies	**20**	**80**	**40**	**80**	**20**	**0**	
Hen	**1**	**2**	**2**	**0**	**1**	**0**	
Boy	**1**	**2**	**2**	**0**	**0**	**2**	
2nd Boy	**1**	**2**	**2**	**0**	**0**	**2**	
Monkey	**1**	**4**	**2**	**0**	**1**	**4**	
Master	**1**	**2**	**2**	**0**	**0**	**2**	

(It is important that all zeros in this exercise be filled in. For though zero is not anything, negation is a mental act, the cognition of a fact. It is important that children should learn to cognize the fact and deal rightly w th it, at an early age.

The seventh column is to be filled in, in each case, according to the decision of the class after discussion of the probabilities of that case.

Exercises such as the above can be varied indefinitely.

They are of great importance; they fix attention on the questions of relevance, and of cumulation or neutralization (e. g. the advent of the master adds to the number of legs, is irrelevant to questions about hoofs or tails, and diminishes the amount of mischief). They call attention to the line of demarcation between certain, calculable knowledge (number of heads, &c.), and probable knowledge, or that which is contingent on facts not yet known, or not ascertainable (the character of the boys). They also call attention to the fact that an additional element may either add its own bulk to that previously existing (heads, tails), or may raise that previously existing to a higher power. A boy once suggested to me that, if two boys were in a field, they would not only do mischief themselves but would start the ponies trying to break fences; and added:—'One boy alone might not think of it, but two together would be *bound* to.' I affirm that the boy's mathematical insight must have been increased by his having made such a suggestion. All such ideas are better suggested by examples of the above kind, where no intellectual work is involved in the question itself and the mind is free for new conceptions to surge up of themselves from the 'abysmal depths of consciousness,' than by explanations thrust in *ab extra* while the child's mind is occupied in struggling with the difficulties of a problem in Arithmetic or Algebra.

For town children, playground may be substituted for field, dogs and cats for ponies).

Mind-Picture.

Shut your eyes, &c. See a field; see ponies, two boys, and monkey. Monkey jumps on the back of a pony and frightens it. Boys shout and scamper. Ponies become wild and break fences. Hen runs away scared and falls into pond, &c., &c., &c.

Master comes in and restores order.

XIV

EXERCISE ON ZERO

Shut eyes, &c. Make a mind-picture :—Me lifting the chalk to the black-board. I make one stroke and then put my hand down. I do this action three times ; how many strokes will be on the board ?

If, instead of making one stroke on the board, I made two and put my hand down ; how many strokes would be on the board when I had done the action once ? Twice ? Three times ? Four times ? Before I had done it at all ?

Open eyes, sit up. Shut eyes, &c. Make a mind-picture :—A clean black-board, me holding the chalk and then putting it down, without touching the board ; what would be on the board? Nothing. Now make another picture :—Me making a stroke. Now I rub the stroke out. What is on the board that you now see in your mind ? Nothing. So if I do nothing, or if I make a stroke and rub it out, the result is the same as far as the board is concerned. Were the two ways of getting it the same? No.

That was a mind-picture black-board. Now open eyes and look at the real one. Here is a quite clean board; I make a stroke ; I rub it out. Is it really quite as clean as when it has

been cleaned on purpose for class? No. What do you see on it? A smudge.

Suppose I lifted the chalk to the board, and made a dot and no stroke at all; how many strokes would be on the board when I had done the action three times? Four times? Six times? Nine times? Once? Before I had done it?

One stroke no time?
Two strokes no time?
No stroke three times?
No stroke one time?
No stroke no time?

XV

$$\frac{1}{0} = \infty$$

The idea of fraction is readily introduced by accustoming children, when the concept 'child' is taken as arithmetical unit, to think of that unit as divided into halves, each side being a half; or the unit may be a monkey, each hand representing a quarter. For instance, a common exercise in Arithmetic is the following:—

If each boy is to have two apples, how many shall I want for two boys, three boys, four boys? &c.

The problem might be stated thus:—

If each boy is to have an apple in each hand, how many will be given to two boys? Three

boys? &c. One boy? Half a boy? If each monkey is to have two nuts in each hand, how many will three monkeys have? Two monkeys? One monkey? Half a monkey? A quarter of a monkey? Three quarters of a monkey?

Such questions as the above may sound foolish, because teachers are not yet accustomed to see their importance in the development of arithmetical faculty. Let us once learn to think of the human mind as intended to build up the material sciences round an organic skeleton made of acts of positing the unit, negation, fraction and reconstruction of the broken unit, and our estimate of the relative values of many things in education will undergo rapid change. Every elementary exercise in number should be applied to the concepts unity, zero, one-half, one-quarter, &c. In constructing his tables, the child should be asked 'Twice one?' 'Twice nought?' and 'Twice a half?' as often as he is asked 'Twice three?'

Another useful exercise is the following:—

I have twenty apples on a plate. Boys pass through the room, each coming up for his allowance. If I give one to each boy, how many boys can I serve before my stock is exhausted? If I give one to each half-boy? If I give two to each boy? If I give two to each half-boy? If I give four, five, ten, twenty, to each boy?

Then come down the series.

If I give ten, five, four, two, one, to each boy? If I give *half* an apple to each boy? Half an apple to each half-boy, a quarter-apple to each boy?

The questions should be kept playing round the fraction of apple and fraction of child, till the children see quite clearly through the whole process, and are familiar with the conception that, at the point where the share of each child is one apple, i. e. where the unitary concept of the class 'children' coincides with the unitary concept of the class 'apples,' a crisis is reached; something *happens*; at that point the number of boys is equal to the number of apples; on one side of it the number of boys is less than that of apples; on the other side it is greater.

The series of questions should be repeated with ten apples in the plate; then with forty apples, then the three series (ten, twenty, forty apples) should be intermixed. And as soon as it can be done without confusion, this question should be asked:—

If I give none to each boy, how many can pass through the room before my stock is exhausted?

We thus introduce the true mathematical conception of 'infinity,' free from all that is hazy or doubtful, or which makes a strain on the nerves or imagination; we call attention to the simple fact that no number of boys passing through the room affects the remaining stock of apples in any way; that, when the share of each boy passes from fraction of apple to negation of apple, the relevance of the number of apples on the plate to the number of boys is suddenly broken.

Then ask:—

If the share of each half-boy is none, what is the share of each boy? And how many boys can pass through before the supply is exhausted?

Exercises of this kind should be strictly limited to such simple fractions as are indicated by the two hands of the child and the four hands of the monkey. No

attempt should be made to use them to impart any premature information about the branch of Arithmetic commonly called 'fractions.'

The object of these examples is not to teach 'fractions,' but to supply elementary Arithmetic with that normal thinking-fibre which is, for a human creature, essential to clear ideas about anything, and which is generated, as above stated, by the mutual inter-action of the ideas of unit, negation and fraction.

XVI

EQUIVALENT FRACTIONS

Sit at ease, shut eyes. Make a mind-picture.

Each one of you is to think of some grown-up person that he likes. I am going to call the person 'Mother'; but you may think of Father or Auntie or any one else you like. Now then.

There is a plate on the table, and a knife, and an apple. Mother cuts the apple into two pieces the same size. We call those pieces? Halves. Now Mother cuts each of those pieces into two. We call those pieces halves of halves, or? Quarters. Mother eats one of the quarters, and says you may have the rest. How many quarters are left for you? Three. Two of them are the two halves of one half-

apple; the other quarter is one of the halves of the other half-apple.

Now, what is left for you when some one takes one quarter of an apple and leaves you the rest? Three quarters.

Open eyes, sit up.

BLACK-BOARD.

If a person takes one quarter of anything and leaves me the rest, what I get is three quarters.

Is that right? Do you quite understand it? Now write it in your table-books. Read it out.

Now sit at ease again. Shut eyes. Make a picture. Me with four of you children. Each of you is to think of himself and three others; any three others you like.

There is a table in front of me, and three apples on a plate. I am going to divide those three apples among the four children. If I give one of the apples to each of your friends, what will be left for you? Nothing. Well, that won't do, will it?

We must try a better way than that. I cut an apple into quarters. How many pieces will there be? Four. I give one piece to each of you; what sized piece will each child have? A quarter. What will be left of that apple on

the plate? Nothing. Then I take the second apple and give each child a quarter of it. So now each child will have? Two quarters of apple; and there will be left on the plate one apple and no piece. Now I cut up the third apple and give each child a quarter. And now there is left on the plate? Nothing. So we have shared up the three apples among four children. And the children have all got just the same quantity. What is each child's share? Three quarters of apple. Are you sure you understand? Shall we go over it all again?

(Repeat the paragraph if necessary.)

Open eyes. Sit up.

BLACK-BOARD.

If three apples are shared between four children, the share of each is three quarters of an apple.

Copy into your table-books under the sentence you wrote last. Read both sentences over. You see that if three things are shared between four children, each one's share is the same size as it would have been if there were only one thing, and some one took away one quarter and left him all the rest. What is alike in the two cases? Your share. Only that. Everything else is different. When we

are talking short, we say that 'A quarter of three is equal to, or the same as, three quarters of one.' And we write it this way :—

BLACK-BOARD.

$\frac{3}{4}$ of 1 = $\frac{1}{4}$ of 3.

Copy that in your books.

When this exercise is gone over, another day, the following should be given as a variation :—

I cut up the three apples into quarters. How many pieces are now on the plate? Twelve. And there are four children. How many pieces can I give to each child? Three. And each piece is? A quarter of apple. So each child's share is? Three quarters of apple.

We now have this :—

BLACK-BOARD.

Three quarters of one apple.
A quarter of each of three apples.
A fourth part of twelve quarter-apples.
All are the same size.

Copy that into your books.

If four children are to share one apple, how many quarter-apples does each child get?

If two children are to share two apples, how many quarter-apples does each get?

If two children are to share three apples?
„ „ „ four apples?
„ „ „ five apples?
„ „ „ six apples?
„ „ „ no apple.

Before children are introduced to the practice of 'cancelling' out, in proportion or fraction sums, it would be desirable to introduce them to the idea underlying such processes in some such way as this:—

Suppose I have four apples to share among eight boys, there are two ways in which I can do it. I can cut up all the apples into halves, and give one piece to each boy, or I can give a whole apple to each two boys, and let one of each pair divide the apple belonging to that pair. I have, to begin with, eight boys and four whole apples on a plate. I shall have at the end eight boys each holding a half-apple. One of the ways of doing this is to make first, four groups, each consisting of two boys and an apple. The other way is to cut up all the apples into halves, and give one to each boy. Whichever way I set about it, I get at the last eight boys each holding half an apple.

It is desirable to avoid suggesting to the child's imagination, at any stage of the process, that two boys and an apple is supposed to be in any sense the equivalent of either eight boys and four apples, or one boy and half an apple. Therefore make a clear mind-

picture of the four groups, each consisting of two boys, one of whom is engaged in cutting an apple into halves.

Enter in books (pupil's own part):—

'If four apples are to be shared among eight boys, the share of each boy is the same, whether one man cuts up all the apples, or whether four people each cut one apple in halves.'

All lessons concerning equivalence of fractions and the dividing of numerator and denominator by the same factor should be linked in the children's minds with this fundamental conception of a set of molecules or groups, each consisting of so many boy-units and so many apple-units; it should be shown that what is cancelled out is the number of such groups; and that the number of the groups is rejected for the present, not as being unimportant in itself, but as being irrelevant to the immediate question on hand.

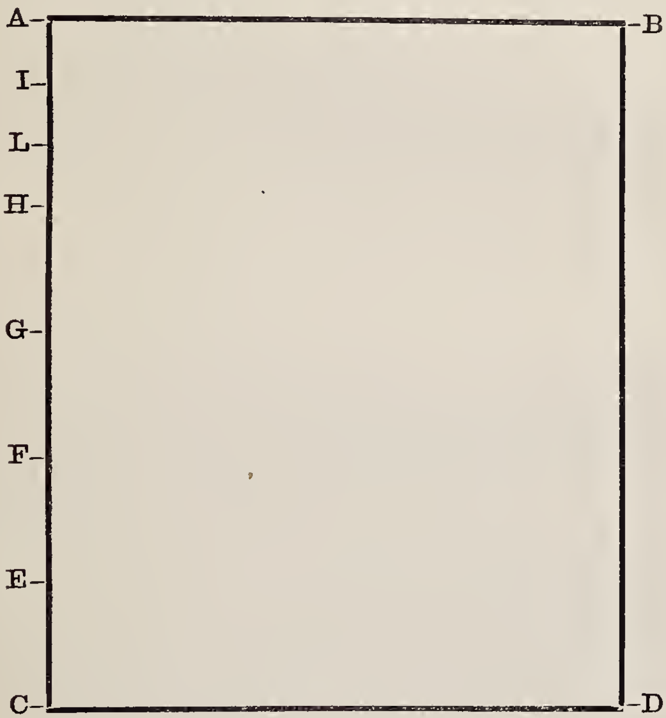
XVII

GREATEST COMMON MEASURE

This is the picture of a ceiling of a room. Suppose a painter wishes to ornament the edge where the ceiling meets the walls, with a little simple pattern that will fit exactly into the lengths of all the sides, without needing any extra corner-ornaments. Suppose he asks us to find out for him the length of the longest

pattern that will fit in exactly. How shall we set about it?

The pattern is to be repeated over and over again. One repeat must end and another begin at A; one must end and another begin at B; the same at C; and also at D.



Let us measure off the length A B from A towards C. We will call the end of that measure E. Some number of repeats will go exactly into A B. We do not yet know what

number of repeats; but some number is to go into A B exactly. The same number will go exactly into A E. E therefore will be one of the points where one pattern ends and another begins. *Some* number of patterns will fit exactly into C E. We must find a length that will fit some number of times into A E, and some other number of times into E C. For the present we can leave A B alone; any pattern that will fit exactly into A E can be copied exactly into A B, because A B is the same length as A E. So now we are going to attend to finding a pattern that will fit exactly into A E and into E C.

Let us measure off the length of E C from E towards A; we will call the end of that measurement F. So F is another point where a pattern begins and ends. Now let us measure off the length C E again, from F towards A; and call the end of that measurement G. Then G is a point where one pattern ends and another begins. We measure off the same length again and call the end H; and again, and call the end I; H and I are points where one pattern ends and another begins.

If we try to measure the length C E from I towards A, we cannot do it; there is not room, I A is not long enough.

Any pattern that will fit into I H will fit also into H G, and G F and F E, and E C; because

all those bits are the same length. So, for the present, we will attend only to finding a length which can repeat some number of times in I A, and some other number of times in I H. Let us measure off the length A I from I towards H. We will call the end of that measurement L. L, then, is one of the points where a pattern ends and another begins. Now we will measure off the same length, A I from L towards H. Ah! now we find that we have come exactly back to H. So that last measurement gives us no new points.

Now let us see what we have got.

A I, I L, and L H, are all the same length. The length of A I has gone twice into I H, therefore it will go into H G twice; and into G F, and F E, and E C twice each. So it fits exactly into A C, and repeats in A C? 11 times, Will it fit exactly into A B? Yes, it fits exactly into A E, and A B is the same length as A E.

Now we will work this off as a sum on another black-board. We will keep this one up, so that we can see what we have been doing.

(If there is no second black-board, the children should be told to work the sum off on paper or slates. On no account must the diagram be rubbed out till the sum has been worked.)

We ticked off the length A B from A towards C; that left E C over. If A B is 18

units and A C is 22 units, how many units is E C ? Four. How do you know that ? Because 18 from 22 = 4. We will put that down as a subtraction sum.

$$\begin{array}{r}22 = \mathbf{A\,C} \\ \underline{18} = \mathbf{A\,E} = \mathbf{A\,B} \\ 4 = \mathbf{E\,C}\end{array}$$

Then we ticked off 4 from the 18.

$$\begin{array}{r}18 = \mathbf{A\,E} \\ \underline{4} = \mathbf{E\,F} \\ 14 = \mathbf{A\,F} \\ \underline{4} = \mathbf{F\,G} \\ 10 = \mathbf{A\,G} \\ \underline{4} = \mathbf{G\,H} \\ 6 = \mathbf{A\,H} \\ \underline{4} = \mathbf{H\,I} \\ 2 = \mathbf{A\,I}\end{array}$$

Then we ticked off

2 = A I = I L from I H

and this left L H without any remainder.

Then we saw that the length A I would measure off exactly, without leaving a remainder, into all the lengths H G, G F, F E, E C, because they are all equal to I H.

Now we will go over that in a different way.

How often will 18 units go into 22 units? Once, leaving a remainder 4.

```
                              18) 22 (1
                                  18
                                  --
Remainder, used as }               4) 18(4
  new divisor      }                  16
                                      --
      Ditto                            2) 4 (2
                                          4
                                       -------
            Remainder                     0
```

Now, will A I fit exactly into A B? What is A B equal to? A E. Well, if the pattern fits into A E, will it fit into A B? And into C D? Yes, because C D is the same length as A B.

And will it fit into B D? Yes. Because? B D is the same length as A C.

You remember that the question we started with was :—What is the *length* of the longest pattern that will fit exactly into all the lengths of the ceiling edge? We have done that. We have done it in three different ways, and we know the longest pattern is 2 units long. We do not know yet how often the pattern will repeat in the room.

How many times will the length A I fit into the length A C?

In A I?	**1**
In I L?	**1**
In L H?	**1**
In H G?	**2**
In G F?	**2**
In F E?	**2**
In E C?	**2**
	11

Eleven times altogether.

How many times in **A B?**	**9**
In B D = A C?	**11**
In C D = A B?	**9**
Answer	**40**

The above exercise repeated occasionally prepares children for understanding G.C.M.; especially if they have learned Division as a shortened method for getting the result of a series of similar subtractions. It may also be given with great advantage to older children who have already been badly taught G.C.M. and become foggy. In their case it should be followed up by a further explanation. Thus:—

'Why could you not understand the rule before? I think it was partly for this reason:—

You have been accustomed to think that the most important part of the answer to a Division sum is the quotient; the remainder matters much less. And when you began to do G.C.M. you were told to do a Division sum; and no

notice was taken of the quotient; you were made to go on as if the remainder were the important thing, and the quotient of no consequence. And that seemed to you somehow dishonest and not quite true; and it puzzled you and prevented your taking in what was really going on. But now you can see. When the painter comes to paint the ceiling border, he must provide paint enough to go all round the room. It would not be honest if he only painted from A to H. If he sends his assistant to block out the points where the patterns must begin and end, the assistant must take the trouble to go all round and divide up the whole edge. If we wanted to find out how many points he will have to mark off, we should have to take account of the quotients and the remainders. Missing a quotient would put us wrong, even more than missing its remainder.

But we did not start to paint the room, nor to block out the edge, nor to find out how many times the pattern would be repeated. We were set to find out, first, what length of pattern would fit in. *For that purpose,* you see, the remainders matter, and the quotients do not.

Now do you see what mistake you made when G.C.M. was being explained to you before?

Mistakes of just that kind are made about many things in life. I am going to tell you

something that may help you not to make them in future.

Sit at ease. Shut eyes. Here is a mind-picture.

There is a village. It has water-works and a reservoir. At the other end of the village there are a few houses which have a deep well of their own. A doctor notices that the people in those houses do not have a kind of illness which is common in the rest of the village. He tells the corporation that he suspects the water of the reservoir is not wholesome somehow, but he does not know exactly what is the matter with it. There is in the village a man who is fond of science, and knows a good deal about it. The corporation ask this man to try to help them to get the water right. He goes to the reservoir. What do you see him doing? Does he slash in pounds and pounds of stuff at random, to try to cure the water before he knows what is wrong with it? No. He has a bottle with him. The bottle holds about a pint. It is perfectly clean, and has a nice clean stopper. He takes a good long look round the reservoir. He then dips the bottle in, fills it, and stoppers it up. He carries it home. He sits down in his study. He puts a drop of the water out of his bottle onto a glass slide, puts the slide under his microscope, and takes a long,

long, careful look at it. He turns a little wheel round, and looks again and again.

Then he puts some water out of his bottle into a small glass, and adds a little of some stuff, and looks, and looks. Then he puts some more in another little glass, and adds a bit of some other stuff, and looks again.

In the middle of all this, two neighbours come in. One says: 'How can you waste your time amusing yourself with your microscope and all these little glasses that are more fit for a doll's house than for a grown man, when your neighbours are suffering and dying because the water is wrong? Why don't you do something to help us?'

The other neighbour says: 'I heard you were employed by the corporation to help the town to get the water pure. Why aren't you at the reservoir doing it? Do you think it is honest to use only a grain of your stuff and purify only a pint-bottle full of water, when you are engaged to cure the reservoir? You will want a hundredweight of disinfectant, not that tiny parcel of it.'

The scientific man answers: 'When we know what is the matter with the water, the corporation will send the right people to cure the evil, and they must take the right amount of stuff, and do the right kind of cleansing. If they

attempt to do with less, that will be dishonest. But at present we do not know what is the matter, nor what treatment the water needs. I have been asked to *find out* these things, and am doing my best to find out. When you are ignorant, experiment on a small scale at first; that is the way to learn.'

Have you taken in that picture? Remember it sometimes, when you are tempted to think that settling something big, in a hurry, before you understand what you are about, will do more good than careful, patient study on a small scale.

(All that is on the two black-boards, to be ultimately entered in form-books.)

XVIII

STANDARD WEIGHTS AND MEASURES

You remember I told you that in teaching a baby to count, it is best to teach him to pack up each ten of his counters or pebbles in a box or heap to itself. You will find many things come easier in your sums, if you take notice that a great deal of what you are learning is to pack things up in your minds, so that they shall be handy and convenient. It would have been inconvenient and troublesome for the shepherds of long ago to make a notch for each sheep that passed through the gateway into the fence; they packed up the number of sheep into tens, and made one stroke—put in the tens' place—stand for ten sheep. Later on, people packed up tens, and made one stroke in the hundreds' place do for ten tens.

In the same way our pennies are packed up into shillings. We need not carry about as many pennies as we shall want to spend; if we want to buy twelve penny eggs, we can pay one little shilling for the lot; because the value of twelve pennies is packed up into a shilling. We do not need to weigh out sixteen separate ounces of lard to make the dough for

sixteen small cakes; sixteen ounce weights are packed into one pound weight.

You will understand sums better if you will remember that much of what you hear about in the Arithmetic-class is a sort of packing of things in your mind or memory.

Things have to be packed in parcels or bundles of different shapes and sizes, according to the nature and use of the things themselves. Let us think of a few different kinds of mind-packing that we know of.

Ordinary numbers are packed in? Tens. And then into? Tens of tens. That plan was started because? Savages found they had ten fingers to count on, before they had names for numbers or knew how to write signs for numbers.

Some things are counted in dozens, such as? Eggs, buttons. Why that is done I really do not know; I have heard that there was once a set of people who had a thumb and five fingers on each hand; that gave them? Twelve counters on their two hands. Perhaps these men invented counting by dozens, but that I cannot say.

Weights go quite differently—at least in England. They are arranged so that each one is twice the weight of the one next below, or, as it is called, they are arranged in *powers of* 2. For some purposes this is a convenient arrange-

ment, and it is a very interesting one to learn about.

Things to drink are arranged in something of the same kind of way; in pints and quarts and gallons. A pint, I believe, is the size that people used to like for their tumblers to drink out of. People in Bavaria still use pint tumblers; it looks very funny to see the children lifting great pint pots to their mouths. We in England have taken to have our tumblers, only half that size; we call a tumbler, half a pint; and we call as much as two tumblers hold, a pint; and what four will hold, a quart; and so on.

Can you tell me how inches are arranged? In twelves. Twelve inches are called a? Foot; though very few men have feet quite twelve inches long. Three twelve-inch feet are called a? Yard. Four inches and a half are the length of an average-sized woman's middle finger; some of you have seen your mothers measuring on their fingers. How many 'fingers' make a yard? Eight.

Some women measure tape or cloth by holding one end of the tape to their noses and then stretching their hands back; that measures off a yard. In old books we read of a *cubit* (that is, the length from the tip of the fingers to the elbow); and a *hand's-breadth*. All these

measurements are rough and not exact, till some Government gives what are called *standard* weights and measures. This saves a good deal of disagreement and quarrelling. People in shops are made to use weights and measures that have been standardized, that is to say, stamped by people appointed by the Government; so that customers can know for certain whether they are getting a full pint, or pound, or yard.

Now when a Government is going to standardize weights or measures, a good deal of thinking has to be done about which are the exact sizes that will pack up into each other most conveniently. You know what I mean by packing, in sums? I do not mean packing actual things in trunks; I mean packing ten ones into a one in the tens' place; and the value of twelve pence in one shilling. Napoleon, more than a hundred years ago, made the French people pack all weights and measures in tens. The way they measure cloth and tapes and roads is this:—Ten centimetres make a decimetre; ten decimetres make a metre; ten metres make a dekametre, and so on. Weights go:—Ten centigrammes make a decigramme; ten decigrammes a gramme, and so on. That is what is called the *decimal* arrangement. We in England have arrangements that

are not so regular; *sixteen* ounces make a pound; *fourteen* pounds make a stone; *twelve* inches make a foot; *three* feet make a yard; and so on. I do not think that either you or I know enough about the matter to be able to judge which plan is best on the whole. Everything costs something in this world; whatever is arranged, something good and nice and interesting will have to be sacrificed to carry it out. The thing of most consequence in the matter is that we should learn to understand other people, and to speak so that they can understand us. We ought to know how they weigh and measure things, and what they mean when they say a pound or a yard; what we are to expect them to give us when we ask for a yard or pound.

If we keep shops by-and-by, we ought to know what we are to give each customer. This is partly why children go to school: to make them understand other people. People used to measure things by 'rule of thumb,' as it is called; every household taking its measures from the length of the father's thumb or the mother's fingers; or the tribe took its measures from the length of the chief's foot. Then when they sold to each other, there were quarrels; because people's feet or fingers did not measure the same length. In civilized countries, people

must have standard weights, and all learn to use the same; so whatever the English Government arrange about weights and measures by-and-by, I hope you will be ready to learn it quickly and use it good-temperedly; and remember that understanding what your neighbours mean and do is of far more consequence than anything you may happen to think, one way or other, about decimal coinage or decimal weights and measures.

And, in order that you may be ready to learn new measures easily and use them cheerfully, you ought to understand a good deal more than most of you do about the different ways in which such things pack up, as we called it. So one day soon we will have a lesson on packing quantities in our sums.

XIX

WHAT CAN BE SETTLED BY HUMAN LAW

On Table:—An ounce weight, a pound weight; a yard tape marked off in inches; a metre ruler marked off in centimetres.

When you see a dozen eggs, or buttons, or stamps, you see the whole twelve in the dozen, and can count them separately.

But when you see a shilling, do you *see* the twelve pennies? No; the pennies are not there at all, only the *cash value* of them is in the shilling.

Here is a pound weight. A pound is? Sixteen ounces. Can you see the ounces separate? No. If I took this to an ironworker's, is there anything he could do to make it show its sixteen ounce weights? Yes, he could melt it down, divide it into ounces, and stamp each 'one ounce'; then each would be a proper ounce weight, just like this one. If I had sixteen like this one, could they be made into a pound weight? Yes; the ironworker could melt them all down together, and make them into a one pound weight like this one.

Is there anything that any one could do to a shilling to make it into twelve pennies? No. Is there any way that any man could make a shilling out of twelve pennies? No.

So you see there are different sorts of packing in our sums. A dozen buttons is twelve buttons; you can see them and count them separately as they lie on their card.

A pound weight can be made into sixteen ounces by melting down and re-stamping; and meantime it already weighs the same as sixteen ounces, and contains the same amount of the same metal.

A shilling could not be made into twelve pennies, and does not weigh them down, and is not the same metal as pennies. But the *value* of twelve pennies is packed into a little bit of a dearer kind of metal.

What is packed into this yard measure? Three feet. What is packed into a foot? Twelve inches. So thirty-six inches are packed in a yard measure. Here is a French metre measure; what is packed into that? A hundred centi-metres.

What are these yard and metre measures most like: dozens, shillings, or weights? They are like dozens, in so far that we can see the inches as they lie along the yard.

Governments can alter the length of the divisions of a yard or metre, in a way that no Government could alter the size of an egg. Eggs grow, inches are only marked off by men.

Men might agree to sell eggs in tens instead of dozens; but they could not alter the size that the eggs grow by merely marking off differently.

What are packed into a year? Months, weeks, days. How many days are packed into a year? Three hundred and sixty-five. Who arranged that? Are there the same number of days in a year in France as here? Could the Government alter that? What makes the length of a day? The sun rising and setting, as we call

it; which means, you know, that the earth turns us to the sun and then away from the sun. What makes the year? Something else that happens between the sun and the earth. The change from day to night and back to day happens three hundred and sixty-five times for once that the other changes happen, the ones that make a year.

Who arranged that? Was it the king? Was it the Parliament? Was it Napoleon or the French Government? Do you think that if all these great people agree together and decided to alter the number of days in a year, they could do it? No. Long ago a number of learned men spent time on *finding out* how many days there are in a year. It took them a very long time to do, but they found out at last. But as for altering the number, oh, no! no one has ever been able to do that.

You see how many different things had to be thought about in arranging the tables in your sum-books. The number of fingers we have; the most convenient way to divide lengths; the most comfortable-sized tumbler to hold; the most convenient coins to carry about; the length of days and years; all sorts of things have been taken into account at different times. So the least we can do is to take things as they come, make the best of them, and learn to understand

Nature, and each other's needs and wishes; and to fit in amongst it all as best we can.

Now sit at ease, shut eyes, and make mind-pictures.

A king sitting in a Parliament House. He says: 'The length of my thumb-joint shall be called an inch; and thirty-six inches shall be called a yard.' London shopkeepers now go to an office at the Guildhall, and take tapes to be marked in inches.

A French general in a cocked hat, with a uniform and sword. He says: 'A metre shall be divided into a hundred centimetres.' French shopkeepers go to an office in Paris, and get their metre tapes measured.

Another picture. Now the king and the general sit waiting. Learned men come and say: 'There *are* three hundred and sixty-five days in a year.' The king and the general bow their heads. The almanack-makers write down what the learned men said [1].

[1] There are 365 days and a few hours in a year. But it is not advisable to enter on details in the present lesson.

XX

PAPER MONEY

On Table :—A £5 note and a £10 note.

I HOPE you have begun to see that Arithmetic is very much concerned with the question of packing things in convenient shapes and sizes, to carry them in our understanding and our memories: sometimes even with convenient packing of things to carry in our hands or pockets. Buttons are sewn in dozens on cards to avoid the need of counting them at busy times, just as grocers put up sugar in pounds ready for customers. Eggs are ready packed in boxes so that a whole dozen can be handled as one boxful. We pack up enough iron or brass to make sixteen ounce weights into one lump, which we call? A pound weight. And so on. If you think, you can see that, if an ancient shepherd made a notch for every ten sheep that had been counted, he was really doing something of the same kind. In the same way we put a mark for a foot at the end of every twelve inches, or a mark for a yard at the end of thirty-six inches.

When we take a shilling in our purse instead of twelve pennies, the packing is of a different

kind. We do not pack the twelve pennies into one lump; we do not take the pennies themselves in any shape. Do you remember what it is that is packed into the little shilling which we use instead of the twelve pennies? The *value* of twelve big clumsy copper pennies is packed into one little bit of the more valuable metal, silver.

In multiplication, when we say 6 times 8 are 48, we pack into a short form the business of adding six eights, one after the other. When we divide 37 by 5, what we are really doing is to see how often we can subtract 5 from 37; instead of doing the subtractions one after another, we pack the whole lot together and do it all in one sentence. If you think about it sometimes, as you go on learning, you will find that a great deal of your Arithmetic is simply a sort of packing up to make short cuts. We get the value, so to speak, of a long roundabout proceeding, packed into a convenient bit of work.

Well now, let us go back to our twelve pence and our shilling. You perfectly understand that the reason why this shilling is worth as much as two sixpences is because there is twice as much silver in it as in one sixpence. It weighs twice as much. And the reason why the silver is worth as much as twelve pennies is because it is a more costly material. Can you

tell me any other instance of one coin being worth a great many others because it is of a more costly material? A sovereign is worth twenty shillings or eight big half-crowns, because gold is much more costly than silver. How many pennies is a sovereign worth? 240. Do you think you quite clearly understand why?

Well now, look at these two pieces of paper. What are they worth? One is worth £5, the other £10. Why? Is paper a more costly material than gold? No. You know that it is very much cheaper by weight. And it would take a great many of such papers as these to weigh down a sovereign. I am sure that these papers are not worth in themselves a penny apiece as paper. And they are exactly alike; that is, the paper part of them is alike; one is not bigger or thicker than the other. Yet I could get five sovereigns for one and ten for the other. Why is that? What makes them so valuable? And why is one worth twice as much as the other? What does it all mean? *Something* must be packed into a little piece of paper, to make it worth so many sovereigns. Let us see if we can find out what is packed into these bits of paper to make them valuable.

It is never worth while guessing at the meaning of a thing until after you have looked

at all that there is to be seen in it, and found out all that there is to learn about it. So we will begin by reading all that is on these papers. And then we shall see if we can make out what it is that has been packed into them.

Bank of England.

I promise to pay the Bearer on Demand the sum of Five Pounds.

London 5 Sept. 1902

For the Gov[rs] and Comp[a] of the
Bank of England.
J. G. Nairne, Chief Cashier.

The Bank of England promises to pay to whoever brings this piece of paper to the Bank, £5. And people feel so *sure* that the Bank will pay the five pounds that anybody will give five pounds for the paper. They feel they will always get their money back.

How did people come to feel so sure that the Bank would keep its promises? Because it has gone on so many generations always keeping them.

In some countries you may see written on a Bank note a promise to pay five dollars, and you may hear people bargaining as to whether that promise (or that bit of paper) is worth two dollars or three. Which means that taking paper money in these countries is more or less a sort of gambling or speculation. People

do not feel sure enough that the promise will be kept to give the full value for the piece of paper that it is written on.

What, then, is it that is packed into these pieces of paper that we call Bank notes? *A habit of trust in the honour of the Bank of England.*

We have been hearing a good deal of late years about the honour of England and the glory of England, and there have been processions and shows and military displays, which are supposed to be for the honour of the country. I should like you to notice, my dears, that any country can wave flags and bang drums and ring bells and walk about in processions and shout about its honour and glory, if it pleases. The real glory and honour of England are much better expressed, it seems to me, by the fact that, if its National Bank promises to pay £5 or £500, the people of the country take that promise for its full value. They believe that their Government will keep its promises.

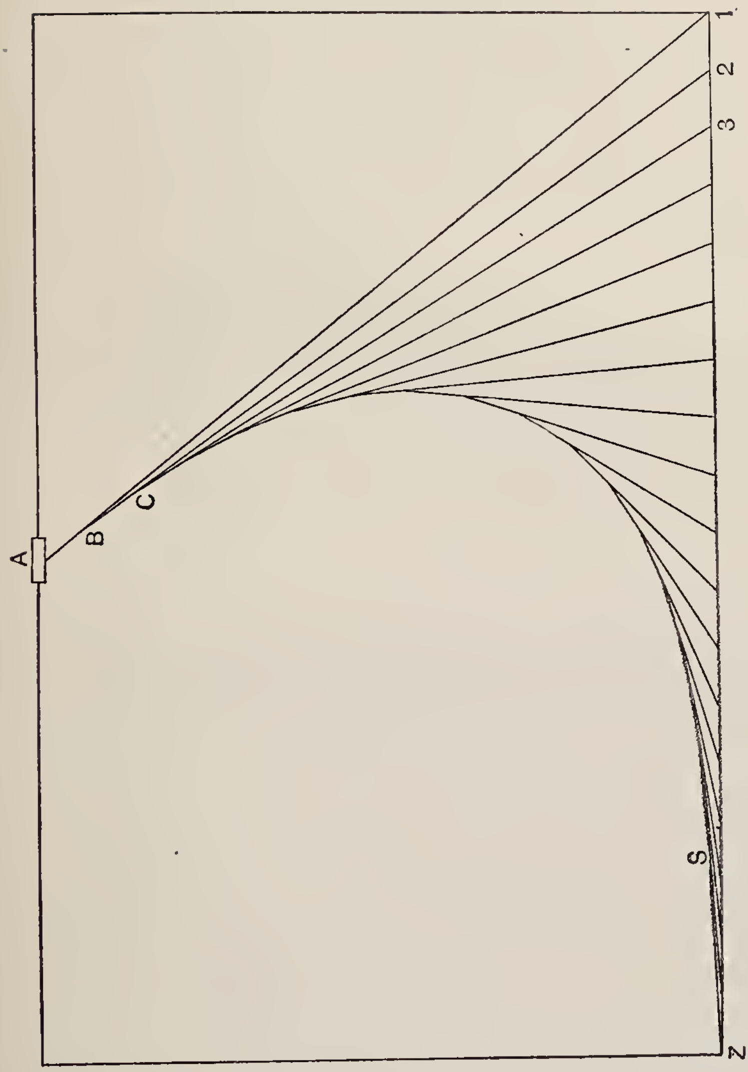
XXI

THE DOG'S PATH

If children are to learn Curve-Tracing, or any sort of Analytical Geometry, the following exercise forms a good introduction to the subject, as it puts their minds into line with the alternation of tendencies, the apparent conflict of forces, by which physical Nature and human evolution are alike worked out. It obviates any sense of mystery about what the whole thing means. Once a child has grown accustomed to pay attention alternately to the aspirations of the dog and to that of the rabbit, and to see a curve growing up under the impulsion of his own alternating sympathy, it is then legitimate to tell him that there are other forces, besides the wishes of animals, at work making curves, and that he can learn to follow the action of many of these, even when he can trace no conscious motive which is setting them to work.

Try to fancy that this black-board is a field. It has a brick wall all round it, with no opening except at A, where there is a gate.

At Z there is a rabbit hole. A rabbit came out into the field at Z and wandered about till he came to here (write the figure 1), where he found something he liked to eat. After a little while, a dog came in at the gate A; the rabbit caught sight of him, and directly afterwards the dog caught sight of the rabbit. Now let us try if we can make out what happened. First we must try to think what the animals would each


1
2
3
A
B
C
S
Z

like to have happen; the rabbit saw the dog first; what do you think he wished? To get away. Perhaps his first idea is to run *straight away* from the dog. But he can't; the wall prevents him. What will he do next? If he was a very foolish rabbit, he might stand trying to get through the wall, and fussing, and saying:—'Oh! dear! I must get out here, there is no other way. Oh! this cruel wicked wall! it is preventing me from getting out! Oh! what *shall* I do? This is the straight way out; and I must knock a hole in this wall! Oh! what *shall* I do?'

If the rabbit was a foolish rabbit, he might go on in that way, till the dog caught him and ate him up.

But we will suppose he is a sensible rabbit who is not fond of knocking his head against brick walls, and who has learned to use his brains properly about his own business. What do you think he will think of next? He will think of trying to get to his hole. He would like to jump straight to his hole; but he cannot go right across a field in one jump. However, we will draw a line to show what he would like to do. (Draw the line 1 Z.)

But just when the rabbit gave up the notion of getting out through the wall at 1, the dog saw him. The dog had his little wishes about

the rabbit. What do you think he wished? What would the dog like to do? Jump on the rabbit and kill him. The dog can no more get across a big field in one jump than the rabbit could; but he would like to do so. We will draw a line to show what the dog thinks he would like to do. (Draw the line A 1.) As the dog cannot jump from A to 1 he jumps as far as he can; his first jump takes him to B. But by the time he has got to B, he sees the rabbit has got to 2. Do you think he will go on scampering down the line A 1 now he sees the rabbit is not at 1, but at 2? Of course not. He will wish now to jump from B to 2. But that again is too far for him to go in one jump; he jumps to C, but by the time he gets there the rabbit is at 3.

The teacher should go on step by step, drawing the successive wishes of the dog, and marking off the jumps of both animals, taking care to keep up, all the time, the children's consciousness of what the animals are each thinking and doing; till the diagram has evolved itself on the black-board. Then:

Now tell me, what do all these lines represent? The line 1 to Z represents the path which the rabbit would like to go along in one jump, and does take in eighteen jumps. The lines A 1, B 2, C 3, and all the other straight lines, each represent a line that the dog at some moment wished to jump along; he jumped

along a bit of one, and then changed his mind and jumped a bit of the next, and so on. We drew all those straight lines; you saw me draw them by the ruler, did you not?

But here is a curved line A to Z. Who drew that? I drew no line except straight ones by the ruler. Look at it well. Make sure that you see it and all the lines on the black-board.

Now sit slack, shut eyes, and think what the curved line is and how it came.

Open eyes and sit up. What is the curved line? The path which the dog really ran, when at each step he meant only to go down some straight line.

XXII

THE BALL'S PATH

You remember that we drew the path of a dog in a field, by drawing straight lines to show what he wished to do but could not. The dog was dragged several ways, first one way and then another. The dog was dragged only by his own wishes and thoughts.

But a thing which has no wishes or thoughts of its own, may be dragged by other forces. We call what makes things move, *force*. Well, as

I said, things may be moved by other forces besides their own thoughts and wishes. If you throw a ball, you make it go along. We do not know that it has any wishes of its own; your wish is a force which moves it. But your wish is not the only force which moves it; for if you aim exactly at the top corner of the house, the ball will not hit exactly the top corner; some other force, what we call its weight, pulls it a little downwards as it goes along; so, between the two forces, the ball makes a curve! When a thing moves in a curve, it is usually because two forces, or more than two, are pulling or pushing it in different directions.

You are going to begin learning about curves. It will help you to keep out of many muddles, if you will try to remember that, when you see a curve in a book or on paper, it represents some real form or movement; or something more or less like a real form or movement. It may be simpler than the real path or shape; but it is *only* simpler; it is really more or less like something meant to be real.

But when you see straight lines, they are seldom meant for anything real. A straight line represents either a path that one force alone *would* have taken something along if no other force had interfered; or else it is just put in for convenience, to measure by. You have per-

haps seen tailors' fashion books, with directions for taking measures. You see a picture of a man with a coat on, and straight lines drawn across the shoulders or bust. You would be dreadfully puzzled if you thought of those lines as parts of the picture; because no coat has lines across the shoulders or bust; but you know they are meant, not for seams in the coat, but for a measuring-tape supposed to be stretched across the man in order to measure the width of his coat. I have known children puzzled out of their wits, and never able to understand their Geometry for years of their school-time, because they mistook straight lines for real parts of some curved thing.

And if you ever feel worried and puzzled over your Geometry lessons, shut your eyes for a minute and think about the rabbit and the dog. Then open eyes, and look at your book; and say to yourself, 'Which parts of this picture are meant to be real like the real path of the dog; and which parts are only like the straight lines that teacher drew on the black-board on purpose to make us children understand how the dog came to run in a path in which at first he did not mean to go?'

XXIII

EXERCISE TO PREPARE FOR GENERAL FORMULAE

1 and 3 are? 4. And 5? 9. And 7? 16. And 9? 25.

And so on, adding successively the odd numbers. The results should be entered in a column on the right hand of the black-board, and ultimately on the right hand of the page in the formula-books. On the left hand should be entered successively:—

$1 \times 1 = 1$

$2 \times 2 = 4$, &c., thus:—

$$
\begin{array}{ll}
1 \times 1 = 1 & 1 \\
 & \underline{3} \\
2 \times 2 = 4 & 4 \\
 & \underline{5} \\
3 \times 3 = 9 & 9 \\
 & \underline{7} \\
4 \times 4 = 16 & 16 \\
 & \underline{9} \\
5 \times 5 = 25 & 25 \\
\text{\&c.} & \text{\&c.}
\end{array}
$$

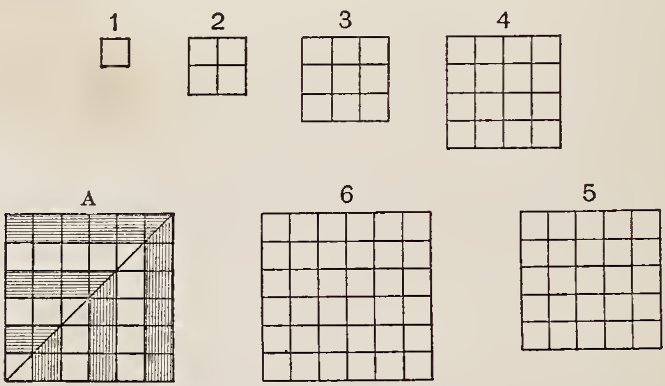
This exercise should be repeated occasionally, as mere practice in adding and multiplying, till the fact of the identity of results of the two processes has been

worked into the children's consciousness, simply as a fact, empirically observed, and, as yet, unexplained.

Then, they should be told to build up the successive odd numbers, either with cubes, or on squared paper (A).

They should be led to see that, by building 3 round 1, they get a square block of the same size and shape as by building 2 on 2; 5 round the block of 4 gives a block the same size and shape as three rows of three, and so on.

These things should be done simply as exercises in adding, multiplying, and neat pencil-shading or chalking. They should each be repeated many times. Great



accuracy should be insisted on; but no explanation should be attempted. The good mathematician will see that none is necessary. An ordinary teacher who attempted one would be almost sure to confuse and mislead. The important thing to secure is that the children's eyes, fingers, and attention should be occupied about the diagrams, thus lodging clear images of them on the memory, ready to crop up on some future occasion; as explained in my Address On the Preparation of the Unconscious Mind for Science [1].

In case the pupil afterwards takes up Algebra or the Calculus, the theory of General (or Sliding) Formulae

[1] *Parent's Review*, 1899.

will, when it has to be explained to him, find ready a substratum of observed and familiar fact, illustrating one such Formula :—

$$(n + 1)^2 = n^2 + 2n + 1$$

The lack of any such basis, the consequent difficulty of making the pupil grasp the idea of a General Formula, forms one of the great obstructions to sound progress in Mathematics.

But, even if he is never to go beyond ordinary Arithmetic, no harm will have been done. He will have gained a little practice, useful in itself, in the comparison of numerical results arrived at by different methods. The time spent on gaining familiarity with the Genesis of Tables of Squares will not have been wasted. Arithmetic itself is always best taught, when it is taught on methods which constitute a sound preparation for the study of the higher Algebra.

APPENDIX

I HAVE not succeeded in drawing out any quite satisfactory scheme for suggesting the feeling of proportion to the imagination of children who have never used any optical instrument. Perhaps some reader may devise one. For children accustomed to use a magnifying-glass, the following mode of conveying the idea is simple. Draw on paper two or three short but unequal lines, and mark each with a letter. Let the children look at each, first with the naked eye, then with a magnifier. Tell them to draw (or to suppose drawn) the lines as seen through the glass. Then say that the little line marked (*a*) is to the big line marked (*a*) in the same proportion as the little line marked (*b*) is to the big line marked (*b*), and as the little line marked (*c*) is to the big line marked (*c*). Draw also a small triangle and quadrilateral, and a copy of each as seen through the magnifier; say that the little triangle is to the big triangle in the same proportion as the little quadrilateral to the big one. Then show (*a*) through a weak magnifier

and (*b*) through a stronger one; and say that, now, the pairs of lines are not in proportion to each other.

I would again emphasize the caution that the lessons in this book are mere specimen types, intended, not to be slavishly read aloud to a class, but to suggest methods, such as I have found useful, of dealing with various arithmetical difficulties. On p. 111 a sum is put in the form kindly shown me by an elementary-school teacher, as the one familiar to the pupils to which she is accustomed. There is a different form more suitable for advanced pupils doing long sums. Each teacher should use the form to which his pupils are accustomed.

On p. 61, the lesson is given as suited to children accustomed to begin multiplying by the unit figure of the multiplier; any teacher who habitually begins at the other end of the multiplier should modify the lesson to suit his class.

When I am giving the lesson on G. C. M., I usually connect it with a story of a painter in Rome, who knew nothing about our system of notation, and who, having to decorate a ceiling, asked his children to spare his labour by marking off, on a plan, as many as they could of the points where he would have to make a pattern

end. This particular story would confuse children not yet accustomed to realize that there have been civilized and artistic people ignorant of our notation; such an idea, suggested for the first time, would distract attention from the arithmetical fact. I have therefore omitted it in the chapter on G. C. M. But I would take this opportunity of reminding teachers that the thought, the imaginative idea, of being engaged in helping a real toiler, has a great effect in checking the tendency to showy guessing, and in fixing attention on the exact nature of the question under consideration. It strengthens the feeling that the object of study is real knowledge and not the mere power to answer examination questions.

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